

Limit Theorems for Sequences

Defn A sequence $\{s_n\}$ is bounded if
 $\exists M \in \mathbb{R}$ such that $|s_n| \leq M \quad \forall n \in \mathbb{N}$.

Note: This definition is equivalent to the set of sequence elements $S := \{s_n : n \in \mathbb{N}\}$ being bounded.

Thm (9.1)

Convergent sequences are bounded.

Thm (9.2-9.6)

Let $\{s_n\}$ and $\{t_n\}$ be sequences that converge to s and t , respectively. Also, let $k \in \mathbb{R}$. Then, we have the following:

- 1) The sequence $\{ks_n\}$ converges to ks . (i.e. $\lim_{n \rightarrow \infty} ks_n = ks$)
- 2) The sequence $\{s_n + t_n\}$ converges to $s+t$. (i.e. $\lim_{n \rightarrow \infty} s_n + t_n = \lim_{n \rightarrow \infty} s_n + \lim_{n \rightarrow \infty} t_n$)
- 3) The sequence $\{s_n t_n\}$ converges to st . (i.e. $\lim_{n \rightarrow \infty} s_n t_n = \left(\lim_{n \rightarrow \infty} s_n\right) \left(\lim_{n \rightarrow \infty} t_n\right)$)
- 4) If $s_n \neq 0 \quad \forall n \in \mathbb{N}$ and $s \neq 0$, then the sequence $\left\{\frac{s_n}{t_n}\right\}$ converges to $\frac{s}{t}$. (i.e. $\lim_{n \rightarrow \infty} \frac{s_n}{t_n} = \frac{\lim_{n \rightarrow \infty} s_n}{\lim_{n \rightarrow \infty} t_n}$)

Notes: 1) This theorem tells us that limits follow the rules of basic algebraic operations ($+$, $-$, $\times$, $\div$), when we don't divide by zero.

2) Along with algebraic manipulation, this theorem and the next enabled you to compute almost all the limits you encountered before in Math 21.

Thm (9.7)

Here is a list of basic limits at your disposal:

1) $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0 \quad \forall p > 0.$

2) $\lim_{n \rightarrow \infty} a^n = 0$ if $|a| < 1$

3) $\lim_{n \rightarrow \infty} n^{1/n} = 1$

4) $\lim_{n \rightarrow \infty} a^{1/n} = 1 \quad \forall a > 0.$

5) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \approx 2.7182818$

Defn

Let $\{s_n\}$ be a sequence.

1) $\{s_n\}$ tends to infinity, written $\lim_{n \rightarrow \infty} s_n = \infty$, if

$$\forall M > 0, \exists N \in \mathbb{R} \text{ such that } \forall n > N \Rightarrow s_n > M. (1)$$

2) $\{s_n\}$ tends to negative infinity, written $\lim_{n \rightarrow \infty} s_n = -\infty$, if

$$\forall M < 0, \exists N \in \mathbb{R} \text{ such that } \forall n > N \Rightarrow s_n < M. (2)$$

3) We say $\{s_n\}$ has a limit (or the limit exists) if $\{s_n\}$ converges or $\{s_n\}$ tends to $\pm\infty$.

Notes: a) The negation of (1) is

$$\exists M > 0, \forall N \in \mathbb{R}, \exists n > N \text{ with } s_n \leq M.$$

b) The negation of (2) is

$$\exists M < 0, \forall N \in \mathbb{R}, \exists n > N \text{ with } s_n \geq M.$$

Thm (9.9)

Let $\{s_n\}$ and $\{t_n\}$ be sequences where $\lim_{n \rightarrow \infty} s_n = \infty$ and $\lim_{n \rightarrow \infty} t_n > 0$.

Then $\lim_{n \rightarrow \infty} s_n t_n = \infty$.

Note: $\lim_{n \rightarrow \infty} t_n$ can be finite or $+\infty$.

Thm (9.10)

Let $\{s_n\}$ be a sequence of positive numbers (i.e. $s_n > 0 \quad \forall n \in \mathbb{N}$).

Then, $\lim_{n \rightarrow \infty} s_n = \infty$ if and only if $\lim_{n \rightarrow \infty} \frac{1}{s_n} = 0$.