Math 25
Vogler

Metric Spaces

Defn

Let S be a set and $d: S \times S \to \mathbb{R}$ (ie $d(x,y)$ is a real valued function). Then d is a *metric* or *distance function* if it satisfies

(i) $d(x,x)=0 \ \forall x \in S$ and $d(x,y)>0 \ \forall x,y \in S$ with $x \neq y$.

(ii) $d(x,y) = d(y,x) \ \forall x,y \in S$.

(iii) $d(x,z) \leq d(x,y) + d(y,z) \ \forall x,y,z \in S$.

The set S together with its metric d is called a *metric space*, written as the pair (S,d).

Notes: 1) Condition (ii) is refered to as the *symmetry* property, and condition (iii) is referred to as the *triangle* inequality.

2) Unless otherwise stated, $\mathbb{R}$ is the metric space $(\mathbb{R}, d)$ with $d(x,y)=|x-y|$, which is what we have been studying thus far, and $\mathbb{R}^n$ is the metric space $(\mathbb{R}^n, d)$ with $d(\vec{x},\vec{y}) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$, called the *Euclidean metric*.

3) The metric solidifies the notion of distance between points in our space S, which allows us to define ideas like convergence and open/closed sets. Changing the metric could change these notions.

4) For us, the pair (S,d) always represents a metric space.

Defn

1) A sequence $\{s_n\}$ in (S,d) <u>converges</u> to s if $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $\forall n > N \Rightarrow d(s_n, s) < \varepsilon$.

2) A sequence $\{s_n\}$ in (S,d) is <u>Cauchy</u> if $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $\forall m,n > N \Rightarrow d(s_n, s_m) < \varepsilon$.

3) (S,d) is said to be <u>complete</u> if every Cauchy sequence converges to some point $s \in S$.

Notes: a) Definitions 1) and 2) are just generalizations of the concepts we studied in $\mathbb{R}$ with $d(x,y) = |x-y|$. I encourage you to compare the two and also come up with the negations for these concepts in a general metric space.

b) Thm 10.11 tells us that $\mathbb{R}$ is a complete metric space. The book has a nice discussion (pg. 85) on how this notion of a complete metric space is connected to the Completeness Axiom.

c) When $\{s_n\}$ converges to s, we still write $\lim_{n\to\infty} s_n = s$.

<u>Notation</u>: When dealing with sequences $\{\vec{x}_k\}$ in $\mathbb{R}^n$, we will write the vector $\vec{x}_k := (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$.

<u>Lemma 13.3</u> A sequence $\{\vec{x}_k\}$ in $\mathbb{R}^n$ converges if and only if each entry $\{x_{k,i}\}$ converges for $i = 1,2,\ldots,n$. The sequence $\{\vec{x}_k\}$ in $\mathbb{R}^n$ is Cauchy if and only if each entry $\{x_{k,i}\}$ is Cauchy for $i = 1,2,\ldots,n$.

<u>Thm 13.4</u>

$\mathbb{R}^n$ is a complete metric space for every $n \in \mathbb{N}$.

<u>Bolzano–Weierstrass Thm for $\mathbb{R}^n$ (13.5)</u>

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Note: Compare this with the Bolzano-Weierstrass Thm (11.5), and the notes there will apply to this theorem.