Math 25
Vogler

Monotone and Cauchy Sequences

Defn
Let $\{s_n\}$ be a sequence of real numbers.
1) $\{s_n\}$ is *increasing* if $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$.
2) $\{s_n\}$ is *decreasing* if $s_n \geq s_{n+1}$ $\forall n \in \mathbb{N}$
3) $\{s_n\}$ is *monotone* (or monotonic) if it's increasing or decreasing.

Note: If $\{s_n\}$ is increasing (or decreasing), then $s_n \leq s_m$ (or $s_n \geq s_m$) when $n < m$.

Thm (10.2)
All bounded monotone sequences converge.

Thm (10.4)
1) If $\{s_n\}$ is an unbounded increasing sequence, then $\lim_{n\to\infty} s_n = \infty$
2) If $\{s_n\}$ is an unbounded decreasing sequence, then $\lim_{n\to\infty} s_n = -\infty$

Note: This fact combined with Thm 10.2 tells us monotonic sequences must go 'somewhere' (either a finite number or $\pm\infty$). Refer to Cor. 10.5 for more details.

Defn

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. Then, we define

1) the limit superior (a.k.a. lim sup) is
$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$

2) the limit inferior (a.k.a. lim inf) is
$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n < N\}$$

Notes: a) One can think of lim sup/lim inf as the supremum/infimum (i.e. least upper bound/greatest lower bound) of ALL 'tails' of the sequence.

b) $\limsup_{n\to\infty} s_n$ might not equal $\sup\{s_n : n\in\mathbb{N}\}$ (i.e. the least upper bound of the set containing all the sequence elements), but we can show $\limsup_{n\to\infty} s_n \le \sup\{s_n : n\in\mathbb{N}\}$. A similar statement applies to lim inf.

c) These concepts will be developed further in Section 12.

Thm (10.7)

Let $\{s_n\}$ be sequence in $\mathbb{R}$.

1) If $\lim_{n\to\infty} s_n = L$, then $\liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L$.

2) If $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$, then $\lim_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$.

Note: L can be finite or $\pm\infty$

Defn A sequence $\{s_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $\forall m,n > N \Rightarrow |s_n - s_m| < \varepsilon$.

Note: The negation is $\exists \varepsilon > 0, \forall N \in \mathbb{R}$ such that $\exists m,n > N$ with $|s_n - s_m| \ge \varepsilon$.

Thm (10.11)

A sequence converges if and only if it is Cauchy.

Note: This is a very useful fact because it allows us to show a sequence converges or diverges without knowing the limit point.