**Math 25**

**Monotone and Cauchy Sequences**

**Defn**

Let \( \{s_n\} \) be a sequence of real numbers.

1) \( \{s_n\} \) is **increasing** if \( s_n \leq s_{n+1} \quad \forall n \in \mathbb{N} \).

2) \( \{s_n\} \) is **decreasing** if \( s_n \geq s_{n+1} \quad \forall n \in \mathbb{N} \).

3) \( \{s_n\} \) is **monotone** (or monotonic) if it's increasing or decreasing.

**Note:** If \( \{s_n\} \) is increasing (or decreasing), then \( s_n \leq s_m \) (or \( s_n \geq s_m \)) when \( n \leq m \).

**Thm (10.2)**

All bounded monotone sequences converge.

**Thm (10.4)**

1) If \( \{s_n\} \) is an unbounded increasing sequence, then

\[
\lim_{n \to \infty} s_n = \infty
\]

2) If \( \{s_n\} \) is an unbounded decreasing sequence, then

\[
\lim_{n \to \infty} s_n = -\infty
\]

**Note:** This fact combined with Thm 10.2 tells us monotonic sequences must go 'somewhere' (either a finite number or \( \pm \infty \)). Refer to Cor. 10.5 for more details.
Defn
Let \( \{S_n\} \) be a sequence in \( \mathbb{R} \). Then, we define

1) the \underline{limit superior} (a.k.a. \( \lim\sup \)) is
\[
\lim\sup_{n \to \infty} S_n = \lim_{N \to \infty} \sup \{ S_n : n > N \}
\]

2) the \underline{limit inferior} (a.k.a. \( \lim\inf \)) is
\[
\lim\inf_{n \to \infty} S_n = \lim_{N \to \infty} \inf \{ S_n : n < N \}
\]

Notes: a) One can think of \( \lim\sup/\lim\inf \) as the supremum/infimum (i.e. least upper bound/greatest lower bound) of \( \text{ALL 'tails'} \) of the sequence.
b) \( \lim\sup_{n \to \infty} S_n \) might not equal \( \sup \{ S_n : n \in \mathbb{N} \} \) (i.e. the least upper bound of the set containing all the sequence elements), but we can show \( \lim\sup_{n \to \infty} S_n \leq \sup \{ S_n : n \in \mathbb{N} \} \). A similar statement applies to \( \lim\inf \).
c) These concepts will be developed further in Section 12.

Thm (10.7)
Let \( \{S_n\} \) be sequence in \( \mathbb{R} \).
1) If \( \lim_{n \to \infty} S_n = L \), then \( \lim\inf_{n \to \infty} S_n = \lim_{n \to \infty} S_n = \lim\sup_{n \to \infty} S_n = L \).
2) If \( \lim\sup_{n \to \infty} S_n = \lim\inf_{n \to \infty} S_n \), then \( \lim_{n \to \infty} S_n = \lim\inf_{n \to \infty} S_n = \lim\sup_{n \to \infty} S_n \).

Note: \( L \) can be finite or \( \pm \infty \)

Defn A sequence \( \{S_n\} \) is a \underline{Cauchy sequence} if
\[ \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m,n > N \implies |S_n - S_m| < \varepsilon. \]

Note: The negation is
\[ \exists \varepsilon > 0, \forall N \in \mathbb{N} \text{ such that } \exists m,n > N \text{ with } |S_n - S_m| \geq \varepsilon. \]

Thm (10.11)
A sequence converges if and only if it is Cauchy.

Note: This is a very useful fact because it allows us to show a sequence converges or diverges without knowing the limit point.