Math 25
Vogler

More on Series

__Defn__ A series $\sum a_k$ satisfies the __Cauchy criterion__ if its sequence of partial sums $\{s_n\}$ is a Cauchy sequence, i.e. $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $\forall m, n > N \Rightarrow |s_n - s_m| < \varepsilon$. (*)

Notes: 1) An equivalent version of (*), which is quite useful, is
$\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $\forall n \geq m > N \Rightarrow \left|\sum_{k=m}^{n} a_k\right| < \varepsilon$.

2) Recall that most series that converge tend to an unknown limit point, which is usually irrational. Thus, the Cauchy criterion is the most common way to show convergence of series in proofs because of its independence from knowing the limit point in advance.

__Thm__ (14.4)
A series converges if and only if it satisfies the Cauchy criterion.

__Cor (14.7)__
If $\sum |a_k|$ converge, then $\sum a_k$ converges.

Note: This states absolutely convergent series must converge.

__Ratio Test__ (Thm 14.8)
Let $\sum a_k$ be series with all nonzero terms. Then,
1) Series converges absolutely if $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$.
2) Series diverges if $\liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$.
3) Otherwise, the test is inconclusive.

Note: Recall, if $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$, then $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$, which gives you the Ratio Test you saw in Math 21 or equivalent.

Root Test (Thm 14.9)

For series $\sum a_k$, let $\alpha = \limsup_{k \to \infty} \sqrt[k]{|a_k|}$. Then,

1) Series converges absolutely if $0 \le \alpha < 1$.
2) Series diverges if $\alpha > 1$.
3) Otherwise, the test is inconclusive if $\alpha = 1$.

Note: In light of Theorem 12.2, the Root Test is stronger than the Ratio Test in the sense of when the Root Test is inconclusive, then the Ratio Test is also inconclusive.

Integral Test (15.2)

For series $\sum a_k$, let $f(k) = a_k \ \forall k \in \mathbb{N}$. Assume $f(x)$ is positive, non-increasing (or decreasing), and continuous for $x \ge 1$

1) If $\int_1^\infty f(x)dx$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2) If $\int_1^\infty f(x)dx$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

Alternating Series Test (Thm 15.3)

Consider a series of the form $\sum (-1)^k a_k$. If the sequence $\{a_k\}$ is positive ($a_n > 0 \ \forall n \in \mathbb{N}$), non-increasing (or decreasing), and $\lim_{k \to \infty} a_k = 0$, then the series $\sum (-1)^k a_k$ converges.

Note: This test cannot be used to show divergence.