Math 25
Vogler

Properties of the Real Numbers

Defn
1) The <u>absolute value</u> of $x \in \mathbb{R}$ is $|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

2) For $a, b \in \mathbb{R}$, the <u>distance between a and b</u> is
$$\text{dist}(a,b) := |a-b|.$$

Thm (3.5)
1) $|a| \geq 0 \quad \forall a \in \mathbb{R}$
2) $|ab| = |a|\,|b| \quad \forall a,b \in \mathbb{R}$
3) $|a+b| \leq |a| + |b| \quad \forall a,b \in \mathbb{R}$
4) $|a-b| \geq ||a|-|b|| \quad \forall a,b \in \mathbb{R}$.

Note: (3) is commonly referred to as the <u>triangle inequality</u>.

Cor 3.6
$$\text{dist}(a,c) \leq \text{dist}(a,b) + \text{dist}(b,c) \quad \forall a,b,c \in \mathbb{R}.$$

Defn Let $S \subseteq \mathbb{R}$ be nonempty.
1) s_0 is the <u>maximum of S</u>, written $s_0 = \max S$, if $s_0 \in S$ and $s \leq s_0 \ \forall s \in S$.
2) s_0 is the <u>minimum of S</u>, written $s_0 = \min S$, if $s_0 \in S$ and $s_0 \leq s \ \forall s \in S$.
3) $M \in \mathbb{R}$ is an <u>upper bound of S</u> if $s \leq M \ \forall s \in S$. Also, we say S is <u>bounded above</u>.
4) $m \in \mathbb{R}$ is a <u>lower bound of S</u> if $m \leq s \ \forall s \in S$. Also, we say S is <u>bounded below</u>.
5) S is <u>bounded</u> if it is bounded above and below.
(i.e. $\exists\, m, M \in \mathbb{R}$ such that $S \subseteq [m, M]$).

Note: The max and min of a set is always an upper and lower bound, respectively.

__Defn__ Let $S \subseteq \mathbb{R}$ be nonempty.

1) $u \in \mathbb{R}$ is the __supremum__ __of__ $\underline{S}$, written $u = \sup S$, if
   a) u is an upper bound of S.
   b) $\forall b \in \mathbb{R}$, if b is an upper bound of S, then $u \leq b$.
2) $\ell \in \mathbb{R}$ is the __infimum__ __of__ $\underline{S}$, written $\ell = \inf S$, if
   a) ℓ is a lower bound of S.
   b) $\forall b \in \mathbb{R}$, if b is a lower bound of S, then $b \leq \ell$.

Note: u is also called the least upper bound of S, and ℓ is also called the greatest lower bound of S.

__Lemma__

1) $u = \sup S$ if and only if
   a) $s \leq u$ $\forall s \in S$. b) Whenever $m < u$, $\exists s \in S$ such that $m < s \leq u$.
2) $\ell = \inf S$ if and only if
   a) $\ell \leq s$ $\forall s \in S$. b) Whenever $\ell < m$, $\exists s \in S$ such that $\ell \leq s < m$.

__Completeness Axiom (4.4)__

Every nonempty subset S of $\mathbb{R}$ that is bounded above has a supremum. (i.e. $\exists u \in \mathbb{R}$ with $u = \sup S$)

__Cor__ (4.5)

Every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum. (i.e. $\exists \ell \in \mathbb{R}$ with $\ell = \inf S$)

__Archimedean Property (4.6)__

If $a > 0$ and $b > 0$, $\exists n \in \mathbb{N}$ such that $na > b$.

__Denseness of $\mathbb{Q}$__ (4.7)

If $a, b \in \mathbb{R}$ and $a < b$, then $\exists r \in \mathbb{Q}$ such that $a < r < b$.

Notes: 1) This tells us that between any two numbers at least one rational number lies between them. Actually, there's an infinite number of them. See HW 4.11.

2) There's also an (infinite number) irrational number between any two numbers. See HW 4.12.