

SequencesDefn

A sequence of real numbers is a function on $\mathbb{N}$, which can be written $f: \mathbb{N} \rightarrow \mathbb{R}$. Usually, we denote an element in the sequence by $s_n := f(n)$ for $n \in \mathbb{N}$, and the whole sequence is written as $\{s_n\}$ or as

$$s_1, s_2, s_3, \dots$$

Notes: 1) Some people refer to an element as a term.

2) Some people start a sequence at $n=0$.

3) Every sequence has a (countably) infinite number of elements.

Defn

A sequence $\{s_n\}$ converges to $s \in \mathbb{R}$, written $\lim_{n \rightarrow \infty} s_n = s$, if

For each $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $n > N \Rightarrow |s_n - s| < \varepsilon$.

Or equivalently,

$$\forall \varepsilon > 0, \exists N \in \mathbb{R} \text{ such that } \forall n > N \Rightarrow |s_n - s| < \varepsilon. (1)$$

If $\{s_n\}$ does not converge, then we say it diverges.

Notes: 1) If $\{s_n\}$ converges, the limit s is unique.

2) I will refer to (1) as the ' εN -property'.

3) Some people force N to be a natural number, which is equivalent because of the Archimedean Property.

4) The negation of (1), which is used to show $\{s_n\}$ diverges, is

$$\exists \varepsilon > 0, \forall N \in \mathbb{R}, \exists n > N \Rightarrow |s_n - s| \geq \varepsilon.$$

5) Condition (1) is an (uncountably) infinite number of statements, one for every ε .

6) $|s_n - s| < \varepsilon$ should be interpreted as the distance between s_n and s is smaller than ε .

Ex Consider the sequence $s_n = \frac{4n^2 + 3n - 2}{2n^2 - 7n - 5}$.

One can show $S = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{4n^2 + 3n - 2}{2n^2 - 7n - 5} = 2$.

Here's a table that shows values for s_n and $\text{dist}(s_n, 2) = |s_n - 2|$ for many values of n .

n	s_n	$ s_n - 2 $
1	-0.5	2.5
2	-1.8181818182	3.8181818182
3	-5.375	7.375
4	-74	76
5	11.3	9.3
6	6.4	4.4
7	4.8863636364	2.8863636364
8	4.1492537313	2.1492537313
9	3.7127659574	1.7127659574
10	3.424	1.424
11	3.21875	1.21875
12	3.0653266332	1.0653266332
13	2.9462809917	0.9462809917
14	2.8512110727	0.8512110727
15	2.7735294118	0.7735294118
16	2.7088607595	0.7088607595
17	2.654185022	0.654185022
18	2.6073500967	0.6073500967
19	2.5667808219	0.5667808219
20	2.5312977099	0.5312977099
21	2.5	0.5
100	2.0885203421	0.0885203421
1000	2.0085338899	0.0085338899
10000	2.0008503376	0.0008503376
100000	2.0000850034	8.50033751391699E-005
1000000	2.0000085	0.0000085
10000000	2.00000085	0.00000085
100000000	2.000000085	0.000000085
1000000000	2.0000000085	8.49999981511473E-009
10000000000	2.0000000009	8.50000070329315E-010
100000000000	2.00000000001	8.50000070329315E-011