Math 25
Vogler

Infinite Series

Defn Given a sequence $\{a_k\}$ of real numbers, the symbol $\sum_{k=0}^{\infty} a_k$ is called an infinite series or simply a series. The symbol is meant to suggest an infinite summation $\sum_{k=0}^{\infty} a_k := a_0 + a_1 + a_2 + a_3 + \cdots$, and each a_k is called a term or element. For all $n \in \mathbb{N}$, let $s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$, which is called the nth partial sum. The resulting sequence $\{s_n\}$ is called the sequence of partial sums.

Notes: 1) We may write $\sum a_k$ instead of $\sum_{k=0}^{\infty} a_k$ when the indexing is understood or irrelevant.
2) While we will usually start a series at $k=0$, we could easily start at any $m \in \mathbb{N}$, in which case the series is $\sum_{k=m}^{\infty} a_k = a_m + a_{m+1} + \cdots$
3) Notice every series has two sequences associated with it, the sequence of terms $\{a_k\}$ and the sequence of partial sums $\{s_n\}$. We will usually focus our attention on the former (i.e. $\{a_k\}$).

Defn The series $\sum a_k$ is said to converge to S, written $\sum a_k = S$, if the sequence of partial sums $\{s_n\}$ converges to S (i.e. $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = S$). If $\{s_n\}$ does not converge, we say the series diverges.

Nth Term Test (Cor. 14.5)

If series $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Note: The contrapositive of this is extremely useful for testing divergence: If $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

__Defn__ For a nonzero constant $a \in \mathbb{R}$, the series $\sum_{k=0}^{\infty} ar^k$ is called a __geometric series__, and the number $r \in \mathbb{R}$ is the __common ratio__ or __ratio__.

__Geometric Series Test__

For $a \neq 0$, the geometric series $\sum_{k=0}^{\infty} ar^k$ converges to the sum $\frac{a}{1-r}$ when $0<|r|<1$, and diverges when $|r| \geq 1$.

__Note__: The geometric series is one of the few series we can easily compute the sum.

__Defn__ For $p \in \mathbb{R}$, the series $\sum_{k=1}^{\infty} \frac{1}{n^p}$ is called a __p-series__

__P-Series Test__

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p>1$ and diverges when $p \leq 1$.

__Defn__ The series $\sum a_k$ is said to be __absolutely convergent__ or __converge absolutely__ if $\sum |a_k|$ converges. If $\sum a_k$ converges and $\sum |a_k|$ diverges, then the series $\sum a_k$ is said to be __conditionally convergent__.

__Comparison Test__ (Thm 14.6)

Let $\sum a_k$ be a series where $a_n \geq 0$ $\forall n \in \mathbb{N}$.

1) If $\sum a_k$ converges and there exists $M \in \mathbb{R}$ with $|b_k| \leq a_k$ $\forall k > M$, then $\sum b_n$ converges absolutely.

2) If $\sum a_k = \infty$ and there exists $m \in \mathbb{R}$ with $a_k \leq |c_k|$ $\forall k > m$, then $\sum c_k$ diverges.

Note: This test is tricky to use because it not only forces you to 'guess' whether the sequence under question converges or diverges, but also relies on your knowledge of convergent and divergent series to compare against.