Math 25

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# Sets on Metric Spaces

<u>Defn</u> Let $x_0 \in S$ and $r>0$. The (open) <u>neighborhood around $x_0$ with radius $r$</u>, denoted $B_r(x_0)$, is the set

$$B_r(x_0) := \{x \in S : d(x, x_0) < r\}$$

Notes: 1) Some people refer to this set as an <u>open ball</u>.
2) These neighborhoods, for any $r>0$, can be viewed as a basis for all open sets (see below) in $S$ and the fundamental open sets in the metric space $(S,d)$.

<u>Defn</u>
Let $E \subseteq S$. An element $x_0 \in E$ is an <u>interior point</u> to $E$ if there exists $r>0$ such that $B_r(x_0) \subseteq E$. We denote the set of all interior points to $E$ as $E^o$. The set $E$ is <u>open</u> if every point in $E$ is an interior point (i.e. $E = E^o$).

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<u>Proposition (13.7)</u> For any metric space $(S,d)$, we have
1) $S$ is open in $S$.
2) The empty set $\emptyset$ is open in $S$.
3) The union of <u>any</u> collection of open sets is open.
4) The intersection of <u>finite many</u> open sets is again an open set.

Note: Any set of points $S$ that satisfies properties (1)-(4) is called a <u>topology</u>, regardless on whether there is a metric on $S$. Topologies are studied in more detail in Math 147.

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<u>Defn</u> A subset $E \subseteq S$ is <u>closed</u> if its compliment $S-E$ is an open set. The <u>closure of $E \subseteq S$</u>, denoted $E^-$, is the intersection of all closed sets containing $E$. The <u>boundary of $E \subseteq S$</u>, denoted $\partial E$, is the set $E^- - E^o$ and the points in $\partial E$ are called <u>boundary points</u>.

Proposition 13.9

Let $E \subseteq S$. Then, we have

a) $E$ is closed if and only if $E = E^-$.

b) $E$ is closed if and only if it contains the limit of every convergent sequence $\{s_n\}$ with $s_n \in E \ \forall n \in \mathbb{N}$.

c) $x \in E^-$ if and only if $\exists \{s_n\}$ with $s_n \in E \ \forall n \in \mathbb{N}$ such that $\lim_{n\to\infty} s_n = x$.

d) $x \in \partial E$ if and only if $x \in E^-$ and $x \in (S-E)^-$.

Note: Some people denote the compliment of E as $E^c := S - E$.

Thm 13.10

Let $\{F_k\}$ be a nested sequence of sets (i.e. $F_{k+1} \subseteq F_k \ \forall k \in \mathbb{N}$) which are all closed bounded nonempty sets of $\mathbb{R}^n$. Then, $F = \bigcap_{k=1}^{\infty} F_k$ is also closed bounded and nonempty.

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Defn

A family $\mathcal{U}$ of open sets is an open cover for a set $E \subseteq S$ if each point in $E$ belongs to at least one set in $\mathcal{U}$ (i.e. $E \subseteq \bigcup\{u : u \in \mathcal{U}\}$)

A subcover of $\mathcal{U}$ is a subfamily of $\mathcal{U}$ which also covers $E$.

A cover or subcover is finite if only contains a finite number of sets, each of which may contain an infinite number of points.

A set $E$ is compact if every open cover of $E$ has a finite subcover of $E$.

Proposition 13.13

Every n-cell in $\mathbb{R}^n$ is compact.

Note: An n-cell $F$ is an $n$-dimensional box $F = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$

In $\mathbb{R}^2$ this is just a rectangle.

Heine-Boral Theorem (13.12)

A set $E \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

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Defn A set $E$ in $\mathbb{R}^n$ is bounded if there exists an $M > 0$ where $\max\{|x_j| : j = 1, \ldots, n\} \leq M \quad \forall \vec{x} = (x_1, x_2, \ldots, x_n) \in E$.