Homework 11 Solutions

10.3) Suppose we are given a decimal expansion $k.d_1d_2d_3d_4...$, where k is a nonnegative integer and $d_j \in \{0,1,2,3,4,5,6,7,8,9\} \quad \forall j \in \mathbb{N}$. Define the corresponding sequence as

$$s_n = k + \frac{d_1}{10} + \frac{d_2}{10^2} + ... + \frac{d_n}{10^n}.$$

Fix $n \in \mathbb{N}$. So, we have

$$s_n = k + \frac{d_1}{10} + \frac{d_2}{10^2} + ... + \frac{d_n}{10^n} \leq k + \frac{9}{10} + \frac{9}{10^2} + ... + \frac{9}{10^n} = k + 1 - \frac{1}{10^n} < k+1,$$

because of the formula given in the Hint. Since $n \in \mathbb{N}$ was arbitrary, we conclude $s_n < k+1 \; \forall n \in \mathbb{N}$.

For the next problem, we need the following fact, which is used in developing the Geometric Series formula (See Example 1 in Section 14):

Proposition 1. *For $n \in \mathbb{N}$ and $a, r \in \mathbb{R}$ with $r \neq 1$, we have the following formula*

$$a(1 + r + r^2 + ... + r^n) = a\left(\frac{1 - r^{n+1}}{1 - r}\right).$$

Notice that the Hint in the above problem (10.3) uses this fact.

10.6) (a) Suppose $\{s_n\}$ is a sequence with $|s_{n+1} - s_n| < 2^{-n} \; \forall n \in \mathbb{N}$. Before we start proving the sequence is Cauchy, we need to get an inequality of only one index n. Notice for $m, n \in \mathbb{N}$ with $m > n$, we have by repeated addition by zero (in a bunch of 'clever' disguises)

$$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + s_{m-2} - s_{m-3} + ... + s_{n+1} - s_n|$$

Then, using the triangle inequality many times, we get

$$|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| ... + |s_{n+1} - s_n| < \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + ... + \frac{1}{2^{n+1}} + \frac{1}{2^n}.$$

Through algebraic manipulation and using Proposition 1 results in

$$|s_m - s_n| < \frac{1}{2^n}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + ... + \left(\frac{1}{2}\right)^{m-n-2} + \left(\frac{1}{2}\right)^{m-n-1}\right) = \frac{1}{2^n}\left(\frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}}\right) = \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}}.$$

Finally, we can obtain the following inequality which will be used to prove the sequence is Cauchy

$$|s_m - s_n| < \frac{1}{2^{n-1}} - \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}},$$

since $m > n$.

Since $n, m \in \mathbb{N}$ with $m > n$ was arbitrary, we have the following

$$|s_m - s_n| < \frac{1}{2^{n-1}} \quad \forall n, m \in \mathbb{N} \text{ with } m > n. \tag{1}$$

Now we move on to proving the sequence is Cauchy. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \dfrac{1}{2^{n-1}} = 0$, $\exists N \in \mathbb{R}$ such that $\forall n > N$, then $\left|\dfrac{1}{2^{n-1}} - 0\right| < \epsilon$. We choose this N. Without loss of generality we assume $m > n$ (otherwise, you just switch the indices). Thus, $\forall m, n > N$, then $|s_m - s_n| < \dfrac{1}{2^{n-1}} < \epsilon$ by equation (1) since $m > n$.

Therefore, we conclude the sequence is Cauchy and must converge by Theorem 10.11.

(b) No. Consider the sequence $s_n = \ln n$. Clearly, $|s_{n+1} - s_n| = \ln n + 1 - \ln n = \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$ $\forall n \in \mathbb{N}$, but $\lim_{n\to\infty} \ln n = \infty$. Since sequence diverges, it cannot be Cauchy by Theorem 10.11.

10.10) Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$ (i.e. a recursion relation).

(a) $s_1 = 1$, $s_2 = \frac{2}{3}$, $s_3 = \frac{5}{9}$, and $s_4 = \frac{14}{27}$.

(b) First, we show $\{s_n\}$ is bounded below by $\frac{1}{2}$ with induction:

(i) This is true for $n = 1$, since $\frac{1}{2} < s_1 = 1$.

(ii) Let $n \in \mathbb{N}$. Suppose that $s_n > \frac{1}{2}$ is true. Then, we have

$$s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}\left(\frac{1}{2} + 1\right) = \frac{1}{2}.$$

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we conclude $\frac{1}{2} < s_n$ $\forall n \in \mathbb{N}$, and the sequence is bounded below.

(c) Second, we show $s_n \geq s_{n+1}$ $\forall n \in \mathbb{N}$ (i.e. sequence is decreasing) by induction:

(i) This is true for $n = 1$, since $s_1 = 1 < \frac{2}{3} = s_2$.

(ii) Let $n \in \mathbb{N}$. Suppose that $s_n \geq s_{n+1}$ is true. Then, we have

$$s_{n+1} = \frac{1}{3}(s_n + 1) \geq \frac{1}{3}(s_{n+1} + 1) = s_{n+2}.$$

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we conclude $s_n \geq s_{n+1}$ $\forall n \in \mathbb{N}$, and the sequence is monotonically decreasing.

(d) By parts (b) and (c) we know $\{s_n\}$ is a bounded monotone sequence, and we conclude it must converge by Theorem 10.2.

Since we know the sequence converges. Let $s = \lim_{n\to\infty} s_n$. Then, from the recursion relation, we have

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{1}{3}(s_n + 1) \Rightarrow s = \frac{1}{3}(s + 1) \Rightarrow 3s = s + 1 \Rightarrow s = \frac{1}{2}.$$

Therefore, we also conclude $\lim_{n\to\infty} s_n = \frac{1}{2}$.

10.11) Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n^2$ for $n \geq 1$ (i.e. a recursion relation).

(a) First, notice $\{t_n\}$ is decreasing since

$$t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n^2 < 1 \cdot t_n = t_n \quad \forall n \in \mathbb{N}.$$

Also, this shows that the sequence is bounded above by 1, since $t_1 = 1$. Second, we show $\{t_n\}$ is bounded below by 0 with induction:

(i) This is true for $n = 1$, since $0 < 1 = t_1$.

(ii) Let $n \in \mathbb{N}$. Suppose that $0 < t_n$ is true. Then, we have

$$4n^2 > 1 \Rightarrow \frac{1}{4n^2} < 1 \Rightarrow 1 - \frac{1}{4n^2} > 0 \Rightarrow t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n^2 > 0.$$

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we have $0 < t_n \quad \forall n \in \mathbb{N}$, and the sequence is bounded.

Therefore, $\{t_n\}$ is a bounded monotone sequence, and we conclude it must converge by Theorem 10.2.

Worksheet 5 Solutions

4) Suppose $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$ with $s_n > 0\ \forall n \in \mathbb{N}$ and $s > 0$. Then, we have the following

$$\lim_{n\to\infty} s_n^{t_n} = \lim_{n\to\infty} e^{t_n \ln s_n} = e^{\lim_{n\to\infty} t_n \ln s_n},$$

by Problem Sheet 4.2 since $f(x) = e^x$ is continuous. Continuing using Theorem 9.4, we have

$$e^{\lim_{n\to\infty} t_n \ln s_n} = e^{(\lim_{n\to\infty} t_n)(\lim_{n\to\infty} \ln s_n)} = e^{t \ln s},$$

by Problem Sheet 4.2 since $f(x) = \ln x$ is continuous. Finally, we get

$$\lim_{n\to\infty} s_n^{t_n} = e^{t \ln s} = s^t,$$

and we conclude if $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$ with $s_n > 0\ \forall n \in \mathbb{N}$ and $s > 0$, then $\lim_{n\to\infty} s_n^{t_n} = s^t$.

5) Suppose E is nonempty subset of $\mathbb{R}$ which is bounded below, and define $L = \{l \in \mathbb{R} : l \text{ is a lower bound for } E\}$.

(a) If $l \in L$, the $l \leq e\ \forall e \in E$. Therefore, any $e \in E$ is an upper bound for L, and $\sup L \in \mathbb{R}$ exists by the Completeness Axiom.

(b) Since $\forall e \in E$ is an upper bound for L, $\sup L \leq e\ \forall e \in E$. Therefore, $\sup L$ is a lower bound for E. Since $l \leq \sup L\ \forall l \in L$, we have $\sup L = \inf E$ by definition.

Remark: This is an alternate proof of Corollary 4.5 of the Completeness Axiom.