Homework 12 Solutions

11.3) (i) For $s_n = \cos\left(\frac{n\pi}{3}\right)$, we have

$$s_n : \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \ldots.$$

(a) Consider the subsequence $\{s_{6k}\}$, so $s_{6k} = 1 \ \ \forall k \in \mathbb{N}$. This subsequence is clearly monotonically increasing. Moreover, it is also monotonically decreasing.

(b) The set of subsequential limits is $S = \left\{\frac{1}{2}, -\frac{1}{2}, -1, 1\right\}$.

(c) By Theorem 11.7, we have

$$\limsup_{n\to\infty} \ s_n = \ \sup S = 1 \text{ and } \liminf_{n\to\infty} \ s_n = \ \inf S = -1.$$

(d) This sequence does not exist (i.e. diverges) by Theorem 11.7 since S has more than one element.

(e) This sequence is bounded since its lim sup and lim inf are both finite.

(ii) For $t_n = \frac{3}{4n+1}$, we have

$$t_n : \frac{3}{5}, \frac{3}{9}, \frac{3}{13}, \frac{3}{17}, \frac{3}{21}, \ldots.$$

(a) Consider the (trivial) subsequence $\{t_k\}$, so $t_k = \frac{3}{4k+1} \ \ \forall k \in \mathbb{N}$. This subsequence is clearly monotonically decreasing.

(b) The set of subsequential limits is $S = \{0\}$.

(c) By Theorem 11.7, we have

$$\limsup_{n\to\infty} \ s_n = \ \sup S = 0 \text{ and } \liminf_{n\to\infty} \ s_n = \ \inf S = 0.$$

(d) This sequence converges to 0 by Theorem 11.7 since S has exactly one element.

(e) This sequence is bounded since its lim sup and lim inf are both finite.

(iii) For $u_n = \left(-\frac{1}{2}\right)^n$, we have

$$u_n : -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \ldots.$$

(a) Consider the (even) subsequence $\{t_{2k}\}$, so $t_{2k} = \frac{1}{2^{2k}}$ $\forall k \in \mathbb{N}$. This subsequence is clearly monotonically decreasing.

(b) The set of subsequential limits is $S = \{0\}$.

(c) By Theorem 11.7, we have

$$\limsup_{n\to\infty} s_n = \sup S = 0 \text{ and } \liminf_{n\to\infty} s_n = \inf S = 0.$$

(d) This sequence converges to 0 by Theorem 11.7 since S has exactly one element.

(e) This sequence is bounded since its lim sup and lim inf are both finite.

(iv) For $v_n = (-1)^n + \frac{1}{n}$, we have

$$v_n : 0, \frac{3}{2}, -\frac{2}{3}, \frac{5}{4}, -\frac{4}{5}, \ldots.$$

(a) Consider the (odd) subsequence $\{t_{2k+1}\}$, so $t_{2k+1} = -1 + \frac{1}{2k+1}$ $\forall k \in \mathbb{N}$. This subsequence is clearly monotonically decreasing.

(b) The set of subsequential limits is $S = \{1, -1\}$.

(c) By Theorem 11.7, we have

$$\limsup_{n\to\infty} s_n = \sup S = 1 \text{ and } \liminf_{n\to\infty} s_n = \inf S = -1.$$

(d) This sequence does not exist (i.e. diverges) by Theorem 11.7 since S has more than one element.

(e) This sequence is bounded since its lim sup and lim inf are both finite.

11.5) Suppose $\{q_n\}$ is the enumeration of all the rational numbers in the interval $(0, 1]$. Let S denote the set of its subsequential limits.

(a) Since every number can be the limit of a subsequence of the enumeration of the rational numbers (see example 11.3), every number in the interval $(0, 1]$ can be the limit of the subsequence of $\{q_n\}$, so we must have $(0, 1] \subset S$. Moreover, since we could easily construct a sequence in $(0, 1]$ that converges to 0 (i.e. $t_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$), 0 must be in S too by Theorem 11.8. Therefore, we conclude $S = [0, 1]$.

(b) By Theorem 11.7, we have

$$\limsup_{n\to\infty} q_n = \sup S = 1 \text{ and } \liminf_{n\to\infty} q_n = \inf S = 0.$$

11.10) Suppose $\{s_n\}$ is the sequence of numbers illustrated in Figure 11.2, listed in the indicated order. Let S denote the set of its subsequential limits.

(a) Notice that the set of terms in the sequence $\{s_n\}$ is

$$E := \{s_n : n \in \mathbb{N}\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ...\right\}$$

Moreover, each term will appear an infinite amount of times since there are an infinite amount of rows in Figure 11.2. So for each each element in E, we could construct a subsequence $\{s_{n_k}\}$ that is identically one of these elements (i.e. $s_{n_k} = \frac{1}{2}\ \ \forall k \in \mathbb{N}$). Also, we can construct a subsequence $\{s_{n_k}\}$ with every value in E (i.e. $s_{n_k} = \frac{1}{k}\ \ \forall k \in \mathbb{N}$), and this subsequence clearly converges to 0. Therefore, we conclude

$$S = E \cup \{0\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, ...\right\}.$$

(b) By Theorem 11.7, we have

$$\limsup_{n\to\infty}\ s_n = \ \sup S = 1 \text{ and } \liminf_{n\to\infty}\ s_n = \ \inf S = 0.$$