# Homework 13 Solutions

For the next problem, we need the following fact which is the result of Exercise 5.4 (an optional homework problem) :

**Proposition 1.** *Let $S \subset \mathbb{R}$ be nonempty. Define $S^- := \{-s : s \in S\}$. Then inf $S = -$sup $S^-$.*

11.8) Suppose we have a sequence $\{s_n\}$. We define the set of terms in the sequence as $E := \{s_n : n \in \mathbb{N}\}$. We consider two cases: (1) $E$ is bounded below or (2) $E$ is not bounded below.

For case (1), suppose $E$ is bounded below. For any $N \in \mathbb{N}$, define $E_N := \{s_n : n > N\}$ and $E_N^- := \{-s_n : n > N\}$. Then by definition, we have

$$\liminf_{n\to\infty}\ s_n := \lim_{N\to\infty}\ \inf E_N = \lim_{N\to\infty}\ -\sup E_N^-,$$

using Proposition 1. By definition again, we continue further obtain

$$\liminf_{n\to\infty}\ s_n := \lim_{N\to\infty}\ \inf E_N = \lim_{N\to\infty}\ -\sup E_N^- = -\lim_{N\to\infty}\ \sup E_N^- =: -\limsup_{n\to\infty}\ -s_n.$$

Thus, for this case we have

$$\liminf_{n\to\infty}\ s_n = -\limsup_{n\to\infty}\ -s_n.$$

For case (1), suppose $E$ is not bounded below. Then, the corresponding sequence $\{s_n\}$ is not bounded below too, and consequently the sequence $\{-s_n\}$ is not bounded above. Hence, we have

$$\liminf_{n\to\infty}\ s_n = -\infty = -\limsup_{n\to\infty}\ -s_n.$$

Therefore, we conclude

$$\liminf_{n\to\infty}\ s_n = -\limsup_{n\to\infty}\ -s_n.$$

11.11) Suppose $S$ is a bounded nonempty set of $\mathbb{R}$. By the Completeness Axiom, sup $S \in \mathbb{R}$ exists. Define $L :=$ sup $S$. There are two cases to consider: (1) $L \in S$ or (2) $L \notin S$.

For case (1), if $L \in S$ (i.e. $L$ is a maximum), we can easily build a monotone sequence that converges to $L$ by defining $s_n = L\ \forall n \in \mathbb{N}$, and we are done.

For case (2), suppose $L \notin S$. We are going to construct an increasing sequence $\{s_n\}$ of points in $S$ (i.e. $s_n \in S\ \forall n \in \mathbb{N}$) such that $lim_{n\to\infty}\ s_n = L$ through construction by induction as follows:

(i) Consider $L - 1 < L$. Since $L =$ sup $S$, $\exists s_1 \in S$ such that $L - 1 < s_1 < L$ because $L \notin S$. So $s_1$ exists.

(ii) Let $n \in \mathbb{N}$. Suppose that

$$L - \frac{1}{j} < s_j < L \text{ and } s_{j-1} \le s_j \text{ for } j = 1, 2, ..., n$$

is true (Note: we do not really need this to construct the next sequence element, and it is here for completeness of the induction hypothesis). Consider $m = \max\{L - \frac{1}{n+1}, s_n\} < L$. Since $L = \sup S$, $\exists s_{n+1} \in S$ such that $m < s_{n+1} < L$ because $L \notin S$. So $s_{n+1}$ exists with the property

$$L - \frac{1}{n+1} < s_{n+1} < L \text{ and } s_n \leq s_{n+1}.$$

Therefore, by the Principle of Complete Induction, we constructed an increasing sequence $\{s_n\}$ with the property

$$L - \frac{1}{n} < s_n < L \quad \forall n \in \mathbb{N}.$$

Since $\lim\limits_{n\to\infty} L - \dfrac{1}{n} = \lim\limits_{n\to\infty} L = L$, we have $\lim\limits_{n\to\infty} s_n = t = L = \sup S$ by the Squeeze Theorem.

Therefore, we conclude if $S$ is a bounded nonempty subset of $\mathbb{R}$, then there exists an increasing sequence $\{s_n\}$ with $s_n \in S$ $\forall n \in \mathbb{N}$ such that $\lim\limits_{n\to\infty} s_n = \sup S$.

## Worksheet 6 Solutions

For the next problem, we need the following inequality:

**Proposition 2.**

$$\frac{1}{m^2} \leq \frac{1}{m(m-1)} = \frac{1}{m-1} - \frac{1}{m} \quad \textit{for } m \geq 2.$$

2) Let

$$s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \quad \forall n \in \mathbb{N}.$$

First notice that the sequence $\{s_n\}$ is clearly increasing. By repeated use of Proposition 2, we have the following inequality

$$s_n = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2} \leq 1+\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+...+\left(\frac{1}{n-1}-\frac{1}{n}\right) = 2-\frac{1}{n} < 2 \;\; \forall n \in \mathbb{N},$$

where a huge amount of cancellation occurs because it's a telescoping sum. So, the sequence is also bounded.

Therefore, we conclude $\{s_n\}$ converges by Theorem 10.2 (which some people call the Monotone Convergence Theorem).

Remark: The is a proof that the series $\sum_{n=1}^{\infty} \frac{1}{n^1}$ converges.

3) (a) Let

$$s_n = \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + ... + \frac{k(n)}{n!} \quad \text{with } k(n) = \frac{1}{3}\left(4\cos\left(\frac{(2n-3)\pi}{3}\right)+1\right) \quad \forall n \in \mathbb{N}.$$

Since all but the last term cancels out, we obtain

$$|s_{n+1} - s_n| = \left| \pm \frac{1}{(n+1)!} \right| = \frac{1}{(n+1)!} \leq \frac{1}{2^n} \quad \forall n \in \mathbb{N},$$

because $(n+1)! = (n+1)\cdot n \cdot (n-1) \cdot ... \cdot 2 \cdot 1 \geq 2 \cdot 2 \cdot 2 \cdot ... \cdot 2 \cdot 1 = 2^n$. Therefore, we conclude that $|s_{n+1} - s_n| \; \forall n \in \mathbb{N}$.

(b) By Homework 10.6 (a), we conclude that $\{s_n\}$ is a Cauchy sequence. Also, $\{s_n\}$ converges by Theorem 10.11.