Thm. The Ratio Test
Let \( \sum a_n \) be a series w/ pos. terms & suppose that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho \)

\[ \Rightarrow \quad \text{a) } \sum a_n \text{ converges if } \rho < 1 \]

\[ \text{b) } \sum b_n \text{ diverges if } \rho > 1 \]

\[ \text{c) } \text{the test is inconclusive if } \rho = 1 \]

pf.

a) \( \rho < 1 \)

Let \( r \) be a \# b/w \( \rho \) \(<1 \) (i.e. \( \rho < r < 1 \)) \( \Rightarrow \ \exists \ \varepsilon = r - \rho > 0 \).

Since \( \frac{a_{n+1}}{a_n} \to \rho \) by defn of limits

\[ \Rightarrow \text{there exists } N \text{ s.t. } n \geq N \Rightarrow \frac{a_{n+1}}{a_n} - \rho < \varepsilon \]

\[ \Rightarrow \frac{a_{n+1}}{a_n} < \rho + \varepsilon = r \text{ when } n \geq N \]

This gives us the following \( \langle \)'s:

\[
\begin{align*}
  a_{N+1} &< ra_N \\
  a_{N+2} &< ra_{N+1} < r^2 a_N \\
  a_{N+3} &< ra_{N+2} < r^3 a_N \\
    \vdots \\
  a_{N+m} &< ra_{N+m-1} < r^m a_N
\end{align*}
\]

\( \text{Consider the series } \sum c_n \text{ w/ } \)

\( c_n = a_n \text{ for } n = 1, 2, \ldots, N \) & \( c_{N+1} = ra_N, \ c_{N+2} = ra_N, \)

\( \ldots \ c_{N+m} = r^m a_N \ldots \)

\( \text{Note that } a_n \leq c_n \text{ for all } n \text{ by construction} \)

& \( \langle \) (*).

Look @ \( \sum c_n = a_1 + a_2 + \ldots + a_{N-1} + a_N + ra_N + r^2 a_N + \ldots \)

\[ = a_1 + a_2 + \ldots + a_{N-1} + a_N \left( 1 + r + r^2 + r^3 + \ldots \right) \]

Geometric Series w/ \( a=1 \)
This geometric series \( \sum_{n=0}^{\infty} r^n \) converges b/c \( |r| < 1 \) \( \Rightarrow \sum_{n=0}^{\infty} c_n \) converges.
Therefore, since \( a_n \leq c_n \) all \( n \), \( \sum a_n \) converges by comparison test.

b) \( p > 1 \)
From some index \( M \) & beyond \( \frac{a_{n+1}}{a_n} > 1 \) (all \( n \geq M \)) \& \( a_M < a_{M+1} < a_{M+2} < \ldots \)
\( \Rightarrow \) the terms of the series don't approach zero as \( n \to \infty \). Hence, the series diverges by the nth-Term test.

c)(\( p = 1 \))
First, consider \( \sum_{i=1}^{\infty} \frac{1}{n} \)
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \frac{n}{n+1} = 1
\]
By the p-series test \( (1 \leq 1) \sum_{i=1}^{\infty} \frac{1}{n} \) diverges.

Second, consider \( \sum_{i=1}^{\infty} \frac{1}{n^2} \)
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \frac{n}{n+1} = 1 \]
By the p-series test \( (2 > 1) \sum_{i=1}^{\infty} \frac{1}{n^2} \) converges.

Hence, when \( p = 1 \) the series can converge or diverge & you need to another test to determine this.