Thm The Ratio Test

Let Σa_n be a series w/ pos. terms & suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$

$\Rightarrow$ a) Σa_n converges if $\rho < 1$

b) Σb_n diverges if $\rho > 1$

c) the test is inconclusive if $\rho = 1$

Pf

a) $\cancel{\$}$ ($\rho < 1$) Let r be a # b/w ρ & 1 (i.e. $\rho < r < 1$) $\Rightarrow \varepsilon = r - \rho > 0$.

Since $\frac{a_{n+1}}{a_n} \to \rho$ by defn of limits

$\Rightarrow$ there exists N s.t. $n \geq N \Rightarrow \frac{a_{n+1}}{a_n} - \rho < \varepsilon$

$\Rightarrow \frac{a_{n+1}}{a_n} < \rho + \varepsilon = r$ when $n \geq N$

This gives us the following <'s:

$$\left.\begin{array}{l} a_{N+1} < r a_N \\ a_{N+2} < r a_{N+1} < r^2 a_N \\ a_{N+3} < r a_{N+2} < r^3 a_N \\ \vdots \\ a_{N+m} < r a_{N+m-1} < r^m a_N \end{array}\right\} (*)$$

Consider the series Σc_n w/

$c_n := a_n$ for $n = 1, 2, \ldots, N$ & $c_{N+1} = r a_N$, $c_{N+2} = r a_N$

.... $c_{N+m} = r^m a_N$ ...

Note that $a_n \leq c_n$ for all n by construction & (*).

Look @ $\sum_1^\infty c_n = a_1 + a_2 + \ldots + a_{N-1} + a_N + r a_N + r^2 a_N + \ldots$

$= a_1 + a_2 + \ldots + a_{N-1} + a_N (\underbrace{1 + r + r^2 + r^3 + \ldots}_{\text{Geometric Series w/ } \substack{r=r \\ a=1}})$

This geometric series $\sum_0^\infty r^n$ converges b/c $|r|<1$ $\Rightarrow$ $\sum_1^\infty c_n$ converges.
Therefore, since $a_n \le c_n$ all n, $\sum a_n$ converges by comparison test.

b) ($\rho > 1$) From some index M & beyond $\frac{a_{n+1}}{a_n} > 1$ (all $n \ge m$) & $a_m < a_{m+1} < a_{m+2} < \dots$
$\Rightarrow$ the terms of the series don't approach zero as $n \to \infty$. Hence, the series diverges by the nth-Term test

c) ($\rho = 1$)
First, consider $\sum_1^\infty \frac{1}{n}$

$$\cancel{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1/(n+1)}{1/n} = r}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{\frac{1}{(n+1)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

By the p-series test ($1 \le 1$) $\sum_1^\infty \frac{1}{n}$ diverges

Second, consider $\sum_1^\infty \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1^2 = 1$$

By the p-series test ($2 > 1$) $\sum_1^\infty \frac{1}{n^2}$ converges

Hence, when $\rho = 1$ the series can converge or diverge & you need to another test to determine this.