<u>Thm</u> Alternating Series Test

$\sum_1^\infty (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$

converges if

1) $a_n > 0$ for all n

2) $a_n \geq a_{n+1}$ for all $n \geq N$ (some int.)

3) $a_n \to 0$

<u>Pf</u>

- If n is even, $n = 2m$,

$\Rightarrow \quad s_n = s_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m})$

$\qquad = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}$

The 1st equality shows $s_{2m+2} \geq s_{2m}$ & the seq $\{s_{2m}\}$ is non increasing b/c $a_n \geq a_{n+1}$.

The 2nd equality shows $s_{2m} \leq a_1$.

So $\{s_{2m}\}$ is non decreasing & bounded from above $\Rightarrow$ convergence to some # L or

$\lim_{m\to\infty} s_{2m} = L \quad (*)$

- If n is odd, $n = 2m+1$,

$\Rightarrow \quad s_{2m+1} = s_{2m} + a_{2m+1}$. ~~Since~~

Since $a_n \to 0$ $\quad \lim_{m\to\infty} a_{2m+1} = 0$

$\Rightarrow \quad \lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + a_{2m+1} = L + 0 = L \quad (\#)$

- Combining $(*)$ & $(\#)$ $\Rightarrow$ $\lim_{n\to\infty} s_n = L$.

Hence, $\sum_1^\infty (-1)^{n+1} a_n = L$ converges

Thm Absolute Convergence Test

If $\sum_1^\infty |a_n|$ converges $\Rightarrow$ $\sum_1^\infty a_n$ converges

Pf

For each n,

$-|a_n| \le a_n \le |a_n| \Rightarrow 0 \le a_n + |a_n| \le 2|a_n|$

If $\sum_1^\infty |a_n|$ converges $\Rightarrow \sum_1^\infty 2|a_n|$

$\Rightarrow$ the non negative series $\sum_1^\infty (a_n + |a_n|)$ converges by the Comparison test

Using one of the favorite tricks of a mathmatician, adding by $0 = |a_n| - |a_n|$

$\Rightarrow a_n = (a_n + |a_n|) - |a_n|$

$\Rightarrow \sum_1^\infty a_n = \sum_1^\infty (a_n + |a_n|) - |a_n| = \sum_1^\infty (a_n + |a_n|) - \sum_1^\infty |a_n|$

There fore, $\sum_1^\infty a_n$ converges.