Section 11.1

2.) $a_n = \frac{1}{n!}$ so $a_1 = \frac{1}{1!} = 1$, $a_2 = \frac{1}{2!} = \frac{1}{2}$,
$a_3 = \frac{1}{3!} = \frac{1}{6}$, $a_4 = \frac{1}{4!} = \frac{1}{24}$

3.) $a_n = \frac{(-1)^{n+1}}{2n-1}$ so $a_1 = \frac{(-1)^2}{1} = 1$,
$a_2 = \frac{(-1)^3}{3} = \frac{-1}{3}$, $a_3 = \frac{(-1)^4}{5} = \frac{1}{5}$,
$a_4 = \frac{(-1)^5}{7} = \frac{-1}{7}$

4.) $a_n = 2 + (-1)^n$ so $a_1 = 2 + (-1) = 1$,
$a_2 = 2 + (-1)^2 = 2+1 = 3$,
$a_3 = 2 + (-1)^3 = 2-1 = 1$,
$a_4 = 2 + (-1)^4 = 2+1 = 3$.

8.) $a_1 = 1$, $a_{n+1} = \frac{a_n}{n+1}$ so
$a_2 = \frac{a_1}{2} = \frac{1}{2}$, $a_3 = \frac{a_2}{3} = \frac{1}{6} = \frac{1}{3!}$,
$a_4 = \frac{a_3}{4} = \frac{1}{24} = \frac{1}{4!}$, $a_5 = \frac{a_4}{5} = \frac{1}{5!}$,
$a_6 = \frac{1}{6!}$, $a_7 = \frac{1}{7!}$, $a_8 = \frac{1}{8!}$, $a_9 = \frac{1}{9!}$,
$a_{10} = \frac{1}{10!}$

11.) $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ so
$a_3 = a_2 + a_1 = 1 + 1 = 2$,
$a_4 = a_3 + a_2 = 2+1 = 3$,
$a_5 = a_4 + a_3 = 3+2 = 5$,
$a_6 = a_5 + a_4 = 5+3 = 8$,

$a_7 = a_6 + a_5 = 8 + 5 = 13$,

$a_8 = a_7 + a_6 = 13 + 8 = 21$,

$a_9 = a_8 + a_7 = 21 + 13 = 34$,

$a_{10} = a_9 + a_8$

12.) $a_1 = 2$, $a_2 = -1$, $a_{n+2} = \frac{a_{n+1}}{a_n}$ so

$a_3 = \frac{a_2}{a_1} = \frac{-1}{2}$,

$a_4 = \frac{a_3}{a_2} = \frac{-1/2}{-1} = \frac{1}{2}$,

$a_5 = \frac{a_4}{a_3} = \frac{1/2}{-1/2} = -1$,

$a_6 = \frac{a_5}{a_4} = \frac{-1}{1/2} = -2$

$a_7 = \frac{a_6}{a_5} = \frac{-2}{-1} = 2$,

$a_8 = \frac{a_7}{a_6} = \frac{2}{-2} = -1$,

$a_9 = \frac{a_8}{a_7} = \frac{-1}{2}$,

$a_{10} = \frac{a_9}{a_8} = \frac{-1/2}{-1} = \frac{1}{2}$

13.) $(-1)^{n+1}$ for $n = 1, 2, 3, \ldots$

16.) $(-1)^{n+1} \cdot \frac{1}{n^2}$ for $n = 1, 2, 3, \ldots$

18.) $-4 + n$ for $n = 1, 2, 3, \ldots$

22.) $\lfloor \frac{n}{2} \rfloor$ for $n = 1, 2, 3, \ldots$

24.) $-1 \leq (-1)^n \leq +1 \rightarrow$

$n-1 \leq n+(-1)^n \leq n+1 \rightarrow$

$\frac{n-1}{n} \leq \frac{n+(-1)^n}{n} \leq \frac{n+1}{n}$; then

$\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right) = 1-0 = 1$ and

$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1+0 = 1$ so

by Sandwich Theorem $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = 1$.

28.) $\lim_{n\to\infty} \frac{n+3}{n^2+5n+6} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{2n+5} = \text{"}\frac{1}{\infty}\text{"} = 0$

29.) $\lim_{n\to\infty} \frac{n^2-2n+1}{n-1} = \lim_{n\to\infty} \frac{(n-1)^2}{n-1}$

$= \lim_{n\to\infty} (n-1) = \infty$ (diverges)

31.) $a_n = 1+(-1)^n$: $0, 2, 0, 2, 0, 2, \ldots$ so

$\lim_{n\to\infty} a_n$ DNE (by oscillation)

35.) $-1 \leq (-1)^{n+1} \leq +1 \longrightarrow$

$\frac{-1}{2n-1} \leq \frac{(-1)^{n+1}}{2n-1} \leq \frac{1}{2n-1}$; then

$\lim_{n\to\infty} \frac{-1}{2n-1} = \text{"}\frac{-1}{\infty}\text{"} = 0$ and

$\lim_{n\to\infty} \frac{1}{2n-1} = "\frac{1}{\infty}" = 0$ so by Sandwich Theorem $\lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1} = 0$.

36.) $\lim_{n\to\infty} \left(\frac{-1}{2}\right)^n = 0$ since $-1 < \frac{-1}{2} < 1$

38.) $\lim_{n\to\infty} \frac{1}{(0.9)^n} = \lim_{n\to\infty} \frac{1}{\frac{9^n}{10^n}}$

$= \lim_{n\to\infty} \frac{10^n}{9^n} = \lim_{n\to\infty} \left(\frac{10}{9}\right)^n = \infty$ (diverges)

since $\frac{10}{9} > 1$.

39.) $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = \sin\left(\frac{\pi}{2} + 0\right)$

$= \sin\frac{\pi}{2} = 1$

42.) $-1 \le \sin n \le +1 \rightarrow 0 \le \sin^2 n \le 1 \rightarrow$

$\frac{0}{2^n} \le \frac{\sin^2 n}{2^n} \le \frac{1}{2^n}$; then

$\lim_{n\to\infty} \frac{0}{2^n} = \lim_{n\to\infty} 0 = 0$ and

$\lim_{n\to\infty} \frac{1}{2^n} = "\frac{1}{\infty}" = 0$ so by Sandwich Theorem $\lim_{n\to\infty} \frac{\sin^2 n}{2^n} = 0$

44.) $\lim_{n\to\infty} \frac{3^n}{n^3} \overset{"\frac{\infty}{\infty}"}{=} \lim_{n\to\infty} \frac{3^n \cdot \ln 3}{3n^2} \overset{"\frac{\infty}{\infty}"}{=}$

$$\lim_{n\to\infty} \frac{3^n \cdot (\ln 3)^2}{6n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{3^n \cdot (\ln 3)^3}{6} = \infty$$

(diverges)

45.) $$\lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{2\sqrt{n}}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1} = \text{"}\frac{1}{\infty}\text{"} = 0$$

48.) $$\lim_{n\to\infty} (0.03)^{1/n} = (0.03)^0 = 1$$

50.) $$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{(-1)}{n}\right)^n = e^{-1}$$

56.) $$\lim_{n\to\infty} (\ln n - \ln(n+1)) \overset{\text{"}\infty-\infty\text{"}}{=} \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)$$

$$= \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right) \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \ln\left(\lim_{n\to\infty} \frac{1}{1}\right) = \ln 1 = 0$$

by continuity of $y = \ln x$

59.) $$a_n = \frac{n!}{n^n}$$

n: 1, 2, 3, 4, 5 ...

$$\frac{n!}{n^n}: \frac{1}{1}, \frac{2\cdot 1}{2\cdot 2}, \frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3}, \frac{4\cdot 3\cdot 2\cdot 1}{4\cdot 4\cdot 4\cdot 4}, \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 5\cdot 5\cdot 5\cdot 5}, \ldots$$

$$= 1, \frac{1}{2}, \left(\frac{2}{3}\right)\left(\frac{1}{3}\right), \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right), \left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{1}{5}\right), \ldots$$

$$\le 1, \frac{1}{2}, \left(\frac{1}{3}\right)^2, \left(\frac{1}{4}\right)^3, \left(\frac{1}{5}\right)^4, \ldots$$

$$\le 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \text{ i.e.,}$$

$0 \le \dfrac{n!}{n^n} \le \dfrac{1}{n}$; then

$\lim\limits_{n\to\infty} 0 = 0 = \lim\limits_{n\to\infty} \dfrac{1}{n}$ so by

Sandwich Theorem $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$.

70.) $\lim\limits_{n\to\infty} \dfrac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n}$

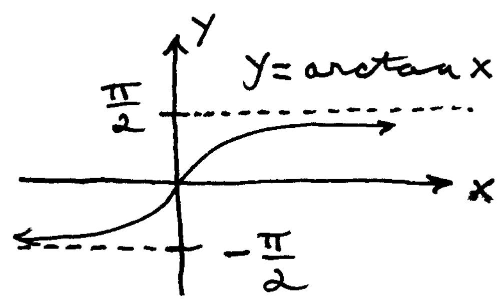
$= \lim\limits_{n\to\infty} \dfrac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n} \cdot \dfrac{\frac{1}{(11/12)^n}}{\frac{1}{\left(\frac{11}{12}\right)^n}}$

$= \lim\limits_{n\to\infty} \dfrac{\left(\frac{120}{121}\right)^n}{\left(\frac{108}{110}\right)^n + 1} = \dfrac{0}{0+1} = 0$

74.) $\lim\limits_{n\to\infty} n\left(1-\cos\frac{1}{n}\right) = \lim\limits_{n\to\infty} \dfrac{1-\cos\frac{1}{n}}{\frac{1}{n}}$

$\overset{\text{"}\frac{0}{0}\text{"}}{=} \lim\limits_{n\to\infty} \dfrac{\sin\frac{1}{n} \cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} = \sin 0 = 0$

75.) $\lim\limits_{n\to\infty} \arctan n = \dfrac{\pi}{2}$



81.) $\lim\limits_{n\to\infty} \left(n - \sqrt{n^2-n}\right) = \text{"}\infty - \infty\text{"}$

$= \lim\limits_{n\to\infty} \left(n-\sqrt{n^2-n}\right) \dfrac{\left(n+\sqrt{n^2-n}\right)}{\left(n+\sqrt{n^2-n}\right)}$

$= \lim\limits_{n\to\infty} \dfrac{n^2-(n^2-n)}{n+\sqrt{n^2-n}} = \lim\limits_{n\to\infty} \dfrac{n}{n+\sqrt{n^2\left(1-\frac{1}{n}\right)}}$

$$= \lim_{n\to\infty} \frac{n}{n+n\sqrt{1-\frac{1}{n}}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}}$$

$$= \frac{1}{1+\sqrt{1-0}} = \frac{1}{2}$$

85.) $x_1 = 1$, $x_{n+1} = x_1 + x_2 + x_3 + \cdots + x_n$, then

$x_2 = x_1 = 1$,

$x_3 = x_1 + x_2 = 1+1 = 2$,

$x_4 = x_1 + x_2 + x_3 = 1+1+2 = 4$,

$x_5 = x_1 + x_2 + x_3 + x_4 = 1+1+2+4 = 8$,

$x_6 = x_1 + x_2 + x_3 + x_4 + x_5 = 1+1+2+4+8 = 16$,

$x_7 = \ldots = 32$, $x_8 = 64$, ...

$x_n = 2^{n-2}$ for $n = 2, 3, 4, 5, \ldots$

94.) Prove that $\lim_{n\to\infty} x^{1/n} = 1$ (for $x>1$):

Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|x^{1/n} - 1| < \varepsilon$.

Then $|x^{1/n} - 1| < \varepsilon$ iff $x^{1/n} - 1 < \varepsilon$ (since $x > 1$)

iff $x^{1/n} < \varepsilon + 1$

iff $\ln x^{1/n} < \ln(\varepsilon+1)$

iff $\frac{1}{n} \ln x < \ln(\varepsilon+1)$

iff $n > \frac{\ln x}{\ln(\varepsilon+1)}$. Let N be any integer $\geq \frac{\ln x}{\ln(\varepsilon+1)}$. Thus, if $n > N$, then $|x^{1/n} - 1| < \varepsilon$. QED

Section 11.2

7.) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = 1 + \frac{-1}{4} + \frac{1}{4^2} + \frac{-1}{4^3} + \frac{1}{4^4} + \frac{-1}{4^5} + \cdots$

$= 1 + \left(\frac{-1}{4}\right) + \left(\frac{-1}{4}\right)^2 + \left(\frac{-1}{4}\right)^3 + \left(\frac{-1}{4}\right)^4 + \cdots$

$= \frac{1}{1-\left(\frac{-1}{4}\right)} = \frac{1}{5/4} = \frac{4}{5}$

8.) $\sum_{n=2}^{\infty} \frac{1}{4^n} = \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \frac{1}{4^5} + \cdots$

$= \frac{1}{4^2} \cdot \left[1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots\right]$

$= \frac{1}{16} \cdot \frac{1}{1-\left(\frac{1}{4}\right)} = \frac{1}{16} \cdot \frac{4}{3} = \frac{1}{12}$

12.) $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right) = 5 \sum_{n=0}^{\infty} \frac{1}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$

$= 5\left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots\right) - \left(1 + \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \cdots\right)$

$= 5 \cdot \frac{1}{1-\left(\frac{1}{2}\right)} - \frac{1}{1-\left(\frac{1}{3}\right)}$

$= 5 \cdot (2) - \frac{3}{2} = \frac{17}{2}$

14.) $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right) = \sum_{n=0}^{\infty} 2 \cdot \frac{2^n}{5^n}$

$= 2 \cdot \left[1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \cdots\right]$

$= 2 \cdot \frac{1}{1-\frac{2}{5}} = 2 \cdot \frac{5}{3} = \frac{10}{3}$

16.) $\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$

$$= \frac{A(2n+1) + B(2n-1)}{(2n-1)(2n+1)} \longrightarrow$$

$A(2n+1) + B(2n-1) = 6$;

Let $x = \frac{1}{2}$: $2A = 6 \rightarrow A = 3$,

Let $x = \frac{-1}{2}$: $-2B = 6 \rightarrow B = -3$; then

$$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \left(\frac{3}{2n-1} + \frac{-3}{2n+1} \right) ;$$

$S_1 = 3 + (-1) = 2$,

$S_2 = (3 + \cancel{(-1)}) + (\cancel{1} + (-3/5)) = 3 - 3/5$,

$S_3 = (3 + \cancel{(-1)}) + (\cancel{1} + \cancel{(\frac{-3}{5})}) + (\cancel{\frac{3}{5}} + (\frac{-3}{7})) = 3 - \frac{3}{7}$,

$S_4 = (3 + \cancel{(-1)}) + (\cancel{1} + \cancel{(\frac{-3}{5})}) + (\cancel{\frac{3}{5}} + \cancel{(\frac{-3}{7})}) + (\cancel{\frac{3}{7}} + (\frac{-3}{9})) = 3 - \frac{3}{9}$,

$\vdots$

$S_n = 3 - \frac{3}{2n+1}$; and by sequence of partial sums series converges since

$$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)} = \lim_{n \to \infty} \left(3 - \frac{3}{2n+1} \right) = 3 - 0 = 3$$

19.) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$;

$S_1 = 1 - \frac{1}{\sqrt{2}}$,

$S_2 = (1 - \cancel{\frac{1}{\sqrt{2}}}) + (\cancel{\frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{3}}) = 1 - \frac{1}{\sqrt{3}}$,

$S_3 = (1 - \cancel{\frac{1}{\sqrt{2}}}) + (\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}) + (\cancel{\frac{1}{\sqrt{3}}} - \frac{1}{\sqrt{4}}) = 1 - \frac{1}{\sqrt{4}}$,

$\vdots$

$s_n = 1 - \frac{1}{\sqrt{n+1}}$; and by sequence of partial sums series converges since

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n+1}}\right)$$

$$= 1 - 0 = 1 .$$

24.) $\sum_{n=0}^{\infty} (\sqrt{2})^n$; $\lim_{n\to\infty} (\sqrt{2})^n = \infty \neq 0$

so series diverges by nth-term test.

25.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3}{2^n} = \sum_{n=1}^{\infty} (-1)^n (-1) \cdot \frac{3}{2^n}$

$= -3 \cdot \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$ so series converges

by geometric series test since $r = -\frac{1}{2}$ and $-1 < r < 1$ with value

$$-3 \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n = -3 \cdot \left[\left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^3 + \cdots\right]$$

$$= (-3)\left(-\frac{1}{2}\right) \cdot \left[1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^3 + \cdots\right]$$

$$= \frac{3}{2} \cdot \frac{1}{1 - \left(\frac{-1}{2}\right)} = \frac{3}{2} \cdot \frac{2}{3} = 1$$

26.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot n$; $\lim_{n\to\infty} (-1)^{n+1} \cdot n \neq 0$

so series diverges by nth-term test.

28.) $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \frac{\cos 0}{1} + \frac{\cos \pi}{5} + \frac{\cos 2\pi}{5^2} + \cdots$

$= 1 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \cdots$

$= 1 + \left(\frac{-1}{5}\right) + \left(\frac{-1}{5}\right)^2 + \left(\frac{-1}{5}\right)^3 + \cdots$

$= \frac{1}{1-(-1/5)} = \frac{5}{6}$ so series <u>converges</u> by geometric series test since $r = -1/5$ and $-1 < r < 1$.

30.) $\sum_{n=1}^{\infty} \ln\left(\frac{1}{n}\right)$; $\lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = -\infty \neq 0$

so series <u>diverges</u> by nth-term test

34.) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$; $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$

$= \lim_{n\to\infty}\left(1+\frac{(-1)}{n}\right)^n = e^{-1} \neq 0$ so

series <u>diverges</u> by nth-term test

36.) $\sum_{n=1}^{\infty} \frac{n^n}{n!} = \frac{1}{1} + \frac{2\cdot 2}{2\cdot 1} + \frac{3\cdot 3\cdot 3}{3\cdot 2\cdot 1} + \frac{4\cdot 4\cdot 4\cdot 4}{4\cdot 3\cdot 2\cdot 1} + \cdots$

$> 1 + 1 + 1 + 1 + \cdots$ so series

<u>diverges</u> by comparison to a divergent series

37.) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} (\ln n - \ln(n+1))$;

$S_1 = \cancel{\ln 1} - \ln 2 = -\ln 2$,

$S_2 = (\cancel{\ln 1} - \cancel{\ln 2}) + (\cancel{\ln 2} - \ln 3) = -\ln 3$,

$S_3 = (\cancel{\ln 1} - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + (\cancel{\ln 3} - \ln 4)$

$= -\ln 4$,

$\vdots$

$S_n = -\ln(n+1)$; and by sequence of partial sums series <u>diverges</u> since

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} -\ln(n+1) = -\infty$$

39.) $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n = 1 + \left(\frac{e}{\pi}\right) + \left(\frac{e}{\pi}\right)^2 + \left(\frac{e}{\pi}\right)^3 + \cdots$

$= \dfrac{1}{1-\left(\frac{e}{\pi}\right)} = \dfrac{\pi}{\pi - e}$ so series

<u>converges</u> by geometric series test since $r = \frac{e}{\pi}$ and $-1 < r < 1$.

52.) $0.234234234\cdots$

$= \dfrac{234}{1000} + \dfrac{234}{1{,}000{,}000} + \dfrac{234}{1{,}000{,}000{,}000} + \cdots$

$= 234\left(\dfrac{1}{10^3} + \dfrac{1}{10^6} + \dfrac{1}{10^9} + \dfrac{1}{10^{12}} + \cdots\right)$

$= (234)\left(\dfrac{1}{10^3}\right) \cdot \left[1 + \dfrac{1}{10^3} + \dfrac{1}{10^6} + \dfrac{1}{10^9} + \cdots\right]$

$$= \frac{234}{1000} \cdot \left[1 + \left(\frac{1}{1000}\right) + \left(\frac{1}{1000}\right)^2 + \left(\frac{1}{1000}\right)^3 + \cdots \right]$$

$$= \frac{234}{1000} \cdot \frac{1}{1 - \frac{1}{1000}} = \frac{234}{999} = \frac{26}{111}$$

53.) $0.7777\cdots$

$$= \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10,000} + \cdots$$

$$= \frac{7}{10} \cdot \left[1 + \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots \right]$$

$$= \frac{7}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{7}{10} \cdot \frac{10}{9} = \frac{7}{9}$$

66.) If $\sum a_n$ converges and $a_n > 0$, then $\lim_{n \to \infty} a_n = 0$. Thus, $\lim_{n \to \infty} \frac{1}{a_n} = "\frac{1}{0^+}" = +\infty \neq 0$ so $\sum \frac{1}{a_n}$ diverges by the nth-term test.

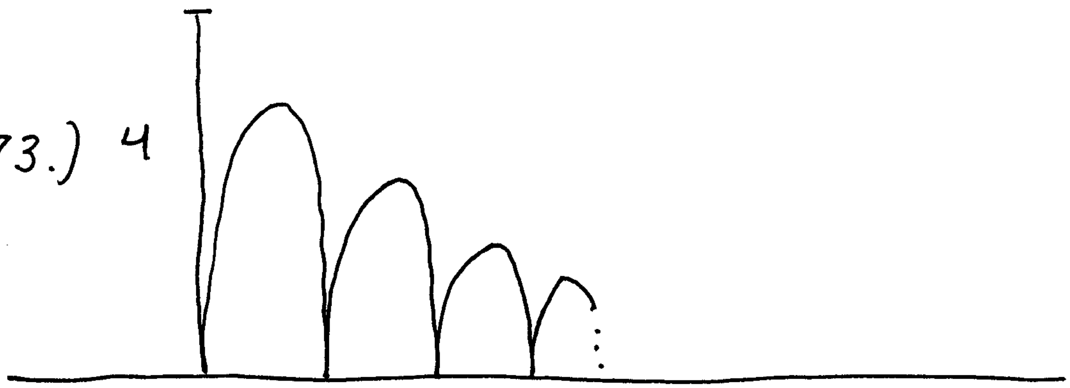
70.) $1 + e^b + e^{2b} + e^{3b} + \cdots = 9 \rightarrow$

$1 + (e^b) + (e^b)^2 + (e^b)^3 + \cdots = 9 \rightarrow$

$\frac{1}{1 - e^b} = 9 \rightarrow \frac{1}{9} = 1 - e^b \rightarrow$

$e^b = \frac{8}{9} \rightarrow b = \ln(8/9)$.

73.)



Total distance is

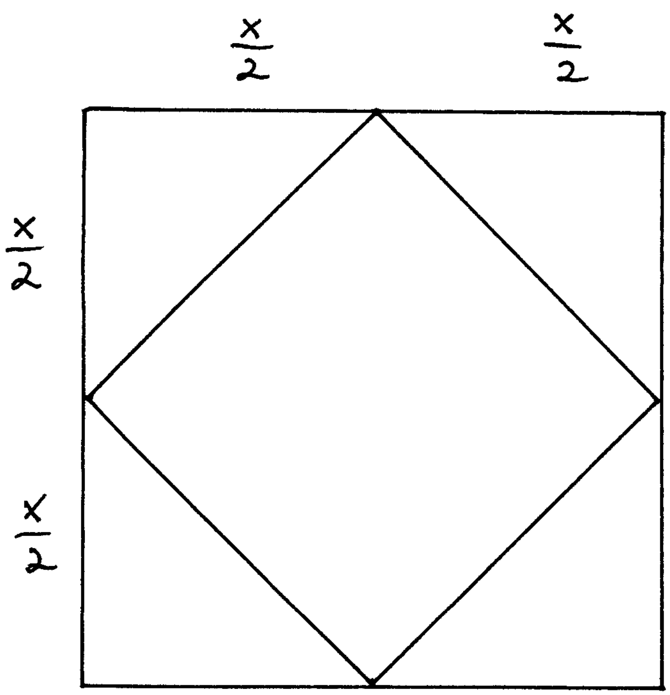
$$T = 4 + 2(0.75)(4) + 2(0.75)^2(4) + 2(0.75)^3(4) + \cdots$$

$$= 4 + (2)(0.75)(4)\cdot\left[1 + (0.75) + (0.75)^2 + \cdots\right]$$

$$= 4 + (6)\cdot\frac{1}{1-(0.75)}$$

$$= 4 + (6)\cdot(4) = 28 \ \cancel{\text{ft}}.$$

75.)



assume big square is $\underline{\underline{x \text{ by } x}}$, then area of inscribed square is

$$\text{Area} = x^2 - 4\left(\frac{x}{2}\right)^2\cdot\frac{1}{2} = \frac{1}{2}x^2;$$

inscribed square is $\underline{\underline{\frac{1}{\sqrt{2}}x \text{ by } \frac{1}{\sqrt{2}}x}}$;

then sum of areas of all squares is

$$S = (2)^2 + \left(\frac{1}{\sqrt{2}} \cdot 2\right)^2 + \left(\left(\frac{1}{\sqrt{2}}\right)^2 \cdot 2\right)^2 + \left(\left(\frac{1}{\sqrt{2}}\right)^3 \cdot 2\right)^2 + \left(\left(\frac{1}{\sqrt{2}}\right)^4 \cdot 2\right)^2 + \cdots$$

$$= 4 + \left(\frac{1}{\sqrt{2}}\right)^2 \cdot (4) + \left(\frac{1}{\sqrt{2}}\right)^4 \cdot (4) + \left(\frac{1}{\sqrt{2}}\right)^6 \cdot (4) + \cdots$$

$$= 4 \cdot \left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right]$$

$$= 4 \cdot \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$= 4 \cdot \frac{1}{1 - \left(\frac{1}{2}\right)} = 8 \ m.^2$$

Section 11.3

3.) $\sum_{n=1}^{\infty} \frac{n}{n+1}$; $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$

$= \frac{1}{1+0} = 1 \neq 0$, so series <u>diverges</u> by <u>nth-term test</u>.

5.) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{3}{n^{1/2}}$; series <u>diverges</u> by <u>p-series</u> test since $p = \frac{1}{2} \leq 1$.

6.) $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$; series <u>converges</u> by <u>p-series</u> test since $p = \frac{3}{2} > 1$.

7.) $\sum_{n=1}^{\infty} \frac{-1}{8^n} = -\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$; series <u>converges</u> by <u>geometric</u> series test since $r = \frac{1}{8}$ and $-1 < r < 1$.

9.) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$; Let $f(x) = \frac{\ln x}{x} \xrightarrow{D}$

$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x}$, so

$+ \quad 0 \quad -$ f'
$x=0 \quad x=e$

f is $+$, $\downarrow$ for $x \geq 3$, and continuous for $x \geq 3$; then

$$\int_3^\infty \frac{\ln x}{x}\,dx = \lim_{A\to\infty} \int_3^A \frac{\ln x}{x}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}(\ln x)^2 \Big|_3^A = \lim_{A\to\infty} \left(\frac{1}{2}(\ln A)^2 - \frac{1}{2}(\ln 3)^2\right)$$

$= \infty$, so series <u>diverges</u> by the <u>integral test</u>.

12.) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$; $\displaystyle\lim_{n\to\infty} \frac{5^n}{4^n+1} = \lim_{n\to\infty} \frac{(5/4)^n}{1+\frac{1}{4^n}}$

$= \dfrac{\infty}{1+0} = \infty \neq 0$, so series <u>diverges</u> by <u>nth-term</u> test.

14.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$; Let $f(x) = \dfrac{1}{2x-1}$, then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

$$\int_1^\infty \frac{1}{2x-1}\,dx = \lim_{A\to\infty} \int_1^A \frac{1}{2x-1}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln|2x-1| \Big|_1^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{2}\ln|2A-1| - \frac{1}{2}\overset{0}{\cancel{\ln 1}}\right) = \infty,$$

so series <u>diverges</u> by <u>integral</u> test.

16.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(\sqrt{n}+1)}$; Let $f(x) = \dfrac{1}{\sqrt{x}\,(\sqrt{x}+1)}$,

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

$$\int_1^{\infty} \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}\,dx = \lim_{A\to\infty} \int_1^{A} \frac{1}{\sqrt{x}\,(\sqrt{x}+1)}\,dx$$

$$= \lim_{A\to\infty} 2\ln|\sqrt{x}+1| \Big|_1^A$$

$$= \lim_{A\to\infty} \left(2\ln|\sqrt{A}+1| - 2\ln 2\right) = \infty,$$

so series <u>diverges</u> by <u>integral test</u>

20.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n} = \sum_{n=1}^{\infty} \left(\frac{1}{\ln 3}\right)^n$ <u>converges</u>

by <u>geometric</u> series test since $r = \dfrac{1}{\ln 3}$ and $-1 < r < 1$.

22.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+(\ln n)^2)}$; Let $f(x) = \dfrac{1}{x(1+(\ln x)^2)}$,

then f is $+$, $\downarrow$, and continuous for $x \geq 1$;

thus, $\displaystyle\int_1^{\infty} \frac{1}{x(1+(\ln x)^2)}\,dx = \lim_{A\to\infty} \int_1^{A} \frac{1}{x(1+(\ln x)^2)}\,dx$

$$= \lim_{A\to\infty} \arctan(\ln x) \Big|_1^A$$

$$= \lim_{A\to\infty} \left(\arctan(\ln A) - \arctan(\overset{0}{\cancel{\ln 1}})\right)$$

$$= \arctan(\infty) - \arctan(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

so series converges by integral test.

23.) $\sum_{n=1}^{\infty} n \cdot \sin\left(\frac{1}{n}\right)$; $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$

$= \lim_{w\to 0} \frac{\sin w}{w} = 1 \neq 0$, so series diverges by nth-term test

25.) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$; Let $f(x) = \frac{e^x}{1+e^{2x}} \xrightarrow{D}$

$$f'(x) = \frac{(1+e^{2x}) \cdot e^x - e^x \cdot 2e^{2x}}{(1+e^{2x})^2}$$

$$= \frac{1-e^{3x}}{(1+e^{2x})^2}$$;

$$\underset{x=0}{\overset{+ \quad 0 \quad -}{\longleftarrow\!\!\!-\!\!\!-\!\!\!\longrightarrow}} \; f'$$

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus, $\int_1^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{A\to\infty} \int_1^A \frac{e^x}{1+(e^x)^2}\,dx$

$= \lim_{A\to\infty} \arctan(e^x)\Big|_1^A$

$= \lim_{A\to\infty} \left(\arctan(e^A) - \arctan(e)\right)$

$= \arctan(\infty) - \arctan(e)$

$= \frac{\pi}{2} - \arctan(e)$, so series converges.

28.) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$; Let $f(x) = \frac{x}{x^2+1}$ $\xrightarrow{D}$

$$f'(x) = \frac{(x^2+1)\cdot(1) - x\cdot(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \ ;$$

then f is $+$, $\downarrow$, and continuous for $x \geq 1$; thus,

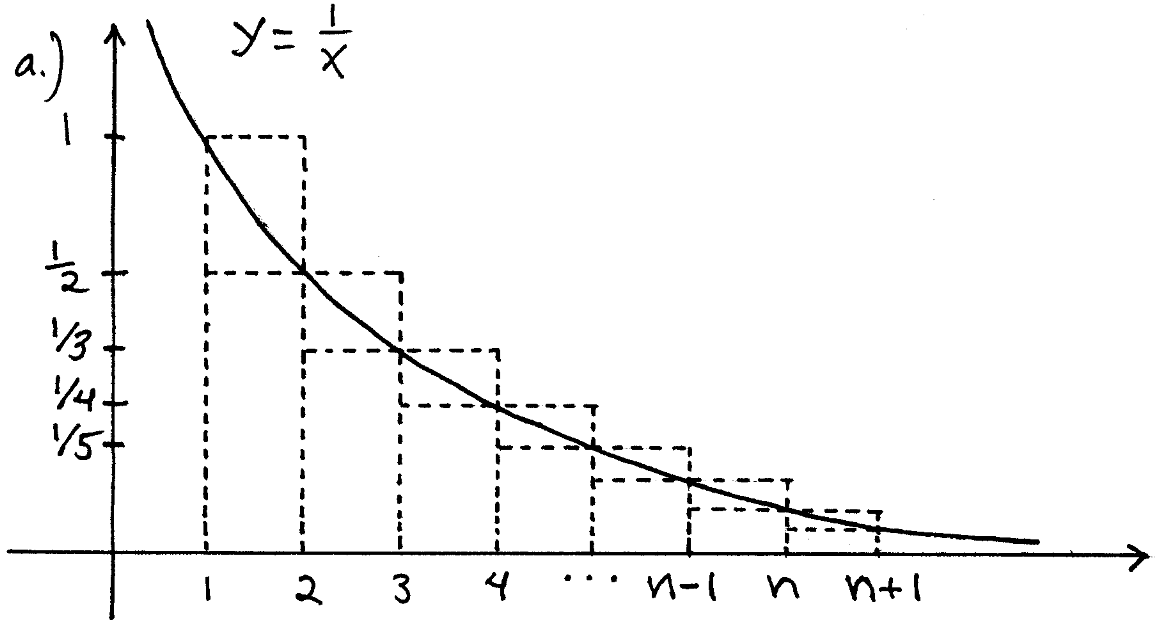
$$\begin{array}{ccc} + & \overset{0}{|} & - \\ & x=1 & \end{array} \quad f'$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{A\to\infty} \int_1^A \frac{x}{x^2+1}\,dx$$

$$= \lim_{A\to\infty} \frac{1}{2}\ln(x^2+1)\Big|_1^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{2}\ln(A^2+1) - \frac{1}{2}\ln 2\right) = \infty ,$$ so series diverges by integral test.

33.) a.)



Consider the integral $\int_1^{n+1} \frac{1}{x}\,dx$

and rectangles <u>above</u> the graph of $y = \frac{1}{x}$ on the interval $[1, n+1]$. Comparing areas results in

$$\int_1^{n+1} \frac{1}{x}\,dx = \ln x \Big|_1^{n+1} = \ln(n+1) - \cancel{\ln 1}^{0}$$

$$\leq \left(\frac{1}{1}\right)(1) + \left(\frac{1}{2}\right)(1) + \left(\frac{1}{3}\right)(1) + \cdots + \left(\frac{1}{n}\right)(1) \text{, i.e.,}$$

(I) $\boxed{\ln(n+1) \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}$;

Consider the integral $\int_1^n \frac{1}{x}\,dx$ and rectangles <u>below</u> the graph of $y = \frac{1}{x}$ on the interval $[1, n]$. Comparing areas results in

$$\left(\frac{1}{2}\right)(1) + \left(\frac{1}{3}\right)(1) + \left(\frac{1}{4}\right)(1) + \cdots + \left(\frac{1}{n}\right)(1) \leq \int_1^n \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^n = \ln n - \cancel{\ln 1}^{0} \text{, i.e.,}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \leq \ln n \rightarrow$$

(II) $\boxed{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 1 + \ln n}$.

b.) $(13{,}000{,}000{,}000 \text{ yrs.})$

$$\times \left(\frac{365 \text{ days}}{\text{yr.}}\right) \times \left(\frac{24 \text{ hrs.}}{\text{day}}\right)$$

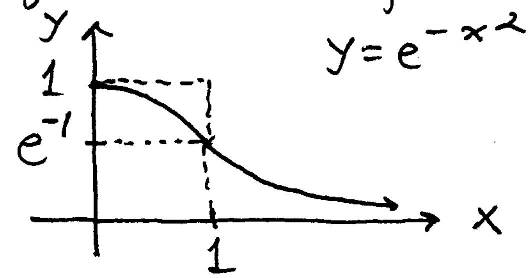
$$\times \left(\frac{60 \text{ min.}}{\text{hr.}}\right) \times \left(\frac{60 \text{ sec.}}{\text{min.}}\right) \approx 4.09968 \times 10^{17} = A \text{ ;}$$

then by (I) and (II)

$$\ln(A+1) \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{A} \le 1 + \ln A \rightarrow$$

$$40.5548 \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{A} \le 41.5548 .$$

42.) $\sum_{n=0}^{\infty} e^{-n^2}$; Let $f(x) = e^{-x^2}$, then f is $+$, $\downarrow$, and continuous for $x \ge 0$; then



$$\int_0^{\infty} e^{-x^2} = \int_0^1 e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx$$

$$\le (1)(1) + \int_1^{\infty} e^{-x}\,dx$$

$$= 1 + \lim_{A \to \infty} \int_1^A e^{-x}\,dx$$

$$= 1 + \lim_{A \to \infty} -e^{-x} \Big|_1^A$$

$$= 1 + \lim_{A \to \infty} \left(\frac{-1}{e^A} - \frac{-1}{e} \right)$$

$$= 1 + \left(0 + \frac{1}{e}\right) = 1 + \frac{1}{e} \ ; \text{ so}$$

$\int_0^{\infty} e^{-x^2}\,dx$ converges.