Section 11.4

2.) $\frac{3}{n+\sqrt{n}} \geq \frac{3}{n+n} = \frac{3}{2n} = \frac{3}{2}\cdot\frac{1}{n}$ and

$\sum_{n=1}^{\infty} \frac{3}{2}\cdot\frac{1}{n} = \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series test since $p = 1 \leq 1$; so

$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ diverges by comparison test

3.) $0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ and

$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges by geometric series test; so $\sum_{n=1}^{\infty}\frac{\sin^2 n}{2^n}$ converges by comparison test

6.) $\sum_{n=1}^{\infty}\frac{n+1}{n^2\cdot\sqrt{n}} = \sum_{n=1}^{\infty}\left(\frac{n}{n^2\sqrt{n}} + \frac{1}{n^2\sqrt{n}}\right)$

$= \sum_{n=1}^{\infty}\frac{1}{n^{3/2}} + \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$; each of these series converges by p-series test since $p = 3/2 > 1$ and $p = \frac{5}{2} > 1$; thus, $\sum_{n=1}^{\infty}\frac{n+1}{n^2\sqrt{n}}$ converges since it is the sum of convergent series

8.) $0 \leq \frac{1}{\sqrt{n^3+2}} \leq \frac{1}{\sqrt{n^3+0}} = \frac{1}{n^{3/2}}$ and

$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges by p-series test since $p = \frac{3}{2} > 1$; so

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+2}}$ converges by comparison test

9.) $\lim_{n\to\infty} \dfrac{\frac{1}{\ln(\ln n)}}{\frac{1}{n}} = \lim_{n\to\infty} \dfrac{n}{\ln(\ln n)}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{1}{\frac{1}{\ln n}\cdot\frac{1}{n}} = \lim_{n\to\infty} n \ln n = \infty$;

since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \geq 1)$, then $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges by limit comparison test

11.) $\lim_{n\to\infty} \dfrac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \dfrac{(\ln n)^2}{n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{2\cdot \ln n \cdot \frac{1}{n}}{1} = \lim_{n\to\infty} \dfrac{2 \ln n}{n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{2\cdot\frac{1}{n}}{1} = 0$; since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$, then $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges by limit comparison test

14.) $\lim_{n\to\infty} \dfrac{\frac{(\ln n)^2}{n^{3/2}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \dfrac{(\ln n)^2}{n^{1/4}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{2\cdot \ln n \cdot \frac{1}{n}}{\frac{1}{4} n^{-3/4}} = \lim_{n\to\infty} 8 \cdot \dfrac{\ln n}{n^{1/4}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} 8 \cdot \dfrac{1/n}{1/4\, n^{-3/4}} = \lim_{n\to\infty} 32 \cdot \dfrac{1}{n^{1/4}} = 0$;

since $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges by p-series test ($p = \frac{5}{4} > 1$), then $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ converges by the limit comparison test

15.) $$\lim_{n\to\infty} \frac{\frac{1}{1+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1+\ln n}$$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty \quad ;$$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test ($p = 1 \leq 1$), then $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ diverges by limit comparison test

19.) $$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2}}{\sqrt{n^2-1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2-1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{1-\left(\frac{1}{n^2}\right)}} = \sqrt{\frac{1}{1-0}} = 1 \quad ;$$

since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by p-series test ($p = 2 > 1$), then $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges by limit comparison test

20.) $\lim_{n\to\infty} \dfrac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \dfrac{n^2}{n^2+1}$

$= \lim_{n\to\infty} \dfrac{1}{1+\frac{1}{n^2}} = \dfrac{1}{1+0} = 1$; since

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series test $(p = \frac{3}{2} > 1)$, then $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges by limit comparison test

22.) $\lim_{n\to\infty} \dfrac{\frac{n+2^n}{n^2 2^n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \dfrac{n+2^n}{2^n}$

$= \lim_{n\to\infty} \left(\frac{n}{2^n} + 1\right) \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \left(\frac{1}{2^n \ln 2} + 1\right)$

$= 0 + 1 = 1$; since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$, then $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$ converges by limit comparison test

23.) $\lim_{n\to\infty} \dfrac{\frac{1}{3^{n-1}+1}}{\left(\frac{1}{3}\right)^n} = \lim_{n\to\infty} \dfrac{3^n}{3^{n-1}+1} \cdot \dfrac{\frac{1}{3^n}}{\frac{1}{3^n}}$

$= \lim_{n\to\infty} \dfrac{1}{\frac{1}{3} + \frac{1}{3^n}} = \dfrac{1}{\frac{1}{3}+0} = 3$; since

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by geometric series test $\left(r = \frac{1}{3}, -1 < r < 1\right)$, then $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1}$ converges by limit comparison test

25.) $\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{n \to \infty} \frac{\cos\left(\frac{1}{n}\right) \cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}}$

$= \cos 0 = 1$; since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \leq 1)$, then $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges by limit comparison test

28.) $\lim_{n \to \infty} \frac{\frac{5n^3 - 3n}{n^2(n-2)(n^2+5)}}{\frac{1}{n^2}}$

$= \lim_{n \to \infty} \frac{(5n^3 - 3n) \cdot \cancel{n^2}}{\cancel{n^2}(n^3 - 2n^2 + 5n - 10)} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}$

$= \lim_{n \to \infty} \frac{5 - 3/n^2}{1 - \frac{2}{n} + \frac{5}{n^2} - \frac{10}{n^3}} = \frac{5 - 0}{1 - 0 + 0 - 0} = 5$;

since $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$, then $\sum_{n=3}^{\infty} \frac{5n^3 - 3n}{n^2(n-2)(n^2+5)}$ converges by limit comparison test

29.) $\lim_{n\to\infty} \dfrac{\frac{\arctan n}{n^{1.1}}}{\frac{1}{n^{1.1}}} = \lim_{n\to\infty} \arctan n$

$= \arctan(\infty) = \frac{\pi}{2}$; since $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges by p-series test ($p = 1.1 > 1$), then $\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.1}}$ converges by limit comparison test

34.) $\lim_{n\to\infty} \dfrac{\frac{n^{1/n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} n^{1/n} = \text{"}\infty^0\text{"}$ (indeterminate)

$= \lim_{n\to\infty} e^{\ln n^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n}$

$= e^{\lim_{n\to\infty} \frac{\ln n}{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim_{n\to\infty} \frac{\frac{1}{n}}{1}} = e^0 = 1$;

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series ($p = 2 > 1$), then $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2}$ converges by limit comparison test

36.) $\sum_{n=1}^{\infty} \frac{1}{1^2 + 2^2 + 3^2 + \cdots + n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{\frac{n(n+1)(2n+1)}{6}} = \sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)}$;

$\lim_{n\to\infty} \dfrac{\frac{6}{n(n+1)(2n+1)}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{6n^3}{n(n+1)(2n+1)}$

$$= \lim_{n\to\infty} 6 \cdot \cancel{\frac{n}{n}} \cdot \frac{n}{n+1} \cdot \frac{n}{2n+1}$$

$$= \lim_{n\to\infty} 6 \cdot \frac{1}{1+\frac{1}{n}} \cdot \frac{1}{2+\frac{1}{n}} = 6 \cdot \frac{1}{1+0} \cdot \frac{1}{2+0}$$

$= \frac{6}{2} = 3$; since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series test ($p = 3 > 1$), then $\sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)}$ converges by limit comparison test

38.) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

<u>Proof</u>: $0 \leq \frac{a_n}{n} \leq a_n$ and $\sum_{n=1}^{\infty} a_n$ converges, so $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by comparison test.

40.) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n^2$ converges.

<u>Proof</u>: Since $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ (by contrapositive of nth-term test). Since

$\lim_{n\to\infty} a_n = 0$ there is some integer N so that $0 \le a_n < 1$ for all $n \ge N$. Thus

$0 \le a_n^2 \le a_n$ for all $n \ge N$

and $\sum_{n=N}^{\infty} a_n$ converges, so

$\sum_{n=N}^{\infty} a_n^2$ converges by comparison test. It follows that

$$\sum_{n=1}^{\infty} a_n^2 = \underbrace{\sum_{n=1}^{N-1} a_n^2}_{\text{finite \#}} + \underbrace{\sum_{n=N}^{\infty} a_n^2}_{\text{finite \#}}$$

converges.

Section 11.5

Using Ratio Test or Root Test:

1.) $\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{\sqrt{2}}}{2^{n+1}}}{\frac{n^{\sqrt{2}}}{2^n}}$

$= \lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{n^{\sqrt{2}}} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} \cdot \left(1+\frac{1}{n}\right)^{\sqrt{2}}$

$= \frac{1}{2} \cdot (1)^{\sqrt{2}} = \frac{1}{2} < 1$, so series converges by ratio test

4.) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{10^n}{10^{n+1}} = \lim_{n\to\infty} (n+1) \cdot \frac{1}{10} = \infty > 1$,

so series diverges by ratio test

5.) $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}}$

$= \lim_{n\to\infty} \frac{(n+1)^{10}}{n^{10}} \cdot \frac{10^n}{10^{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{10} \cdot \frac{1}{10}$

$= (1)^{10} \cdot \frac{1}{10} = \frac{1}{10} < 1$, so series converges by ratio test.

12.) $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$; $\lim_{n\to\infty} \left[\frac{(\ln n)^n}{n^n}\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln n}{n}$

$$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0 < 1$$, so series converges by root test

14.) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$; $\lim_{n\to\infty} \left[\left(\frac{1}{n} - \frac{1}{n^2}\right)^n\right]^{1/n}$

$= \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 - 0 = 0 < 1$, so series converges by root test

18.) $\sum_{n=1}^{\infty} \frac{n^3}{e^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^3}{e^{n+1}}}{\frac{n^3}{e^n}}$

$= \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{e^n}{e^{n+1}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 \cdot \frac{1}{e}$

$= (1)^3 \cdot \frac{1}{e} = \frac{1}{e} < 1$, so series converges by ratio test

19.) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!\, n!\, 3^n}$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+4)!}{3!(n+1)!\,3^{n+1}}}{\frac{(n+3)!}{3!\,n!\,3^n}}$$

$$= \lim_{n\to\infty} \frac{(n+4)!}{(n+3)!} \cdot \cancel{\frac{3!}{3!}} \cdot \frac{n!}{(n+1)!} \cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} (n+4) \cdot \frac{1}{(n+1)} \cdot \frac{1}{3}$$

$$= \lim_{n\to\infty} \frac{1 + \frac{4}{n}}{1 + \frac{1}{n}} \cdot \frac{1}{3} = \frac{1+0}{1+0} \cdot \frac{1}{3} = \frac{1}{3}$$

21.) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(2(n+1)+1)!}}{\frac{n!}{(2n+1)!}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!}$

$= \lim_{n\to\infty} (n+1) \cdot \frac{1}{(2n+3)(2n+2)} = \lim_{n\to\infty} \frac{n+1}{4n^2+10n+6}$

$= \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}}{4+\frac{10}{n}+\frac{6}{n^2}} = \frac{0+0}{4+0+0} = 0 < 1$, so

series converges by ratio test

22.) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$; $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}}$

$= \lim_{n\to\infty} \cancel{(n+1)} \cdot \frac{n^n}{\cancel{(n+1)}(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

$= \lim_{n\to\infty} \left(\frac{1}{\frac{n+1}{n}}\right)^n = \lim_{n\to\infty} \frac{1^n}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$,

so series converges
by ratio test

28.) $a_1 = 1$, $a_{n+1} = \frac{1+\arctan n}{n} \cdot a_n$;

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1+\arctan n}{n} \cdot \cancel{a_n}}{\cancel{a_n}}$

$= \frac{1+\frac{\pi}{2}}{\infty} = 0 < 1$, so series converges
by ratio test

29.) $a_1 = \frac{1}{3}$, $a_{n+1} = \frac{3n-1}{2n+5} \cdot a_n$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3n-1}{2n+5} \cdot \cancel{a_n}}{\cancel{a_n}}$$

$$= \lim_{n\to\infty} \frac{3 - \frac{1}{n}}{2 + \frac{5}{n}} = \frac{3-0}{2+0} = \frac{3}{2} > 1,$$ so

series diverges by ratio test.

37.) $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1} \cdot (n+1)! \cdot (n+1)!}{(2(n+1))!}}{\frac{2^n \cdot n! \cdot n!}{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} 2 \cdot (n+1) \cdot (n+1) \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} 2 \frac{(n^2+2n+1)}{4n^2+6n+2} = \lim_{n\to\infty} \frac{\cancel{2}}{\cancel{2}} \cdot \frac{n^2+2n+1}{2n^2+3n+1}$$

$$= \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{2 + \frac{3}{n} + \frac{1}{n^2}} = \frac{1+0+0}{2+0+0} = \frac{1}{2} < 1,$$

so series converges by ratio test

40.) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$; apply root test :

(*) $$\lim_{n\to\infty} \left[\frac{(n!)^n}{n^{(n^2)}}\right]^{1/n} = \lim_{n\to\infty} \frac{n!}{n^n} = \boxed{?}$$;

this is a sneaky twist: consider the $\underline{\underline{\text{series}}}$ $\sum_{n=1}^{\infty} \frac{n!}{n^n}$;

apply ratio test: (SEE Problem 22)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \cdots = \frac{1}{e} < 1$$

so series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges;

thus, $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (by contrapositive to nth-term test);

(*) so $\lim_{n\to\infty} \left[\frac{(n!)^n}{n^{(n^2)}}\right]^{1/n} = \lim_{n\to\infty} \frac{n!}{n^n} = 0 < 1$,

so series $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$ converges by root test

43.) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n 2^n n!}$;

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2(n+1)-1)}{4^{n+1}2^{n+1}(n+1)!}}{\frac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n 2^n n!}}$$

$$= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1)}{1\cdot 3\cdot 5\cdots(2n-1)} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{2^n}{2^{n+1}} \cdot \frac{n!}{(n+1)!}$$

$= \lim_{n\to\infty} (2n+1) \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{n+1}$

$= \lim_{n\to\infty} \frac{1}{8} \cdot \frac{2n+1}{n+1} = \lim_{n\to\infty} \frac{1}{8} \cdot \frac{2+\frac{1}{n}}{1+\frac{1}{n}}$

$= \frac{1}{8} \cdot \frac{2+0}{1+0} = \frac{1}{4} < 1$, so series converges by ratio test

<u>Using Any Test</u>:

6.) $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$; $\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{(-2)}{n}\right)^n$
$= e^{-2} = \frac{1}{e^2} \neq 0$, so series diverges by nth-term test

7.) $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$; $0 \le \frac{2+(-1)^n}{1.25^n} \le \frac{2+1}{1.25^n} = 3\left(\frac{4}{5}\right)^n$;
$\sum_{n=1}^{\infty} 3\left(\frac{4}{5}\right)^n$ converges by geometric series test ($r = \frac{4}{5}$, $-1 < r < 1$), so $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$ converges by comparison test

8.) $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$; $r = -\frac{2}{3}$, $-1 < r < 1$, so series converges by geometric series test

10.) $\sum_{n=1}^{\infty} \left(1-\frac{1}{3n}\right)^n$; $\lim_{n\to\infty} \left(1-\frac{1}{3n}\right)^n = \lim_{n\to\infty} \left(1+\frac{(-1/3)}{n}\right)^n$

$= e^{-1/3} = \frac{1}{e^{1/3}} \neq 0$ so series diverges by nth-term test

11.) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$; $\lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln n}{n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by

p-series test $(p = 2 > 1)$, so

$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by limit comparison test

13.) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n^2}$

diverges by subtle facts about series, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \leq 1)$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$.

15.) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$; $\lim_{n\to\infty} \frac{\frac{\ln n}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \ln n$

$= \infty$; $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \leq 1)$, so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

diverges by limit comparison test

23.) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$; $\lim_{n\to\infty} \left[\frac{n}{(\ln n)^n}\right]^{1/n}$

$= \lim_{n\to\infty} \frac{n^{1/n}}{\ln n}$ (and $\lim_{n\to\infty} n^{1/n} = $ "∞^0"

(indeterminate) $= \lim_{n\to\infty} e^{\ln n^{1/n}}$

$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim_{n\to\infty} \frac{\ln n}{n}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim_{n\to\infty} \frac{1/n}{1}} = e^0 = 1.$)

$= \frac{1}{\infty} = 0 < 1$, so series converges by root test

Section 11.6

1.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2}$; Let $a_n = \frac{1}{n^2}$, then a_n is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so

series converges by alternating series test

3.) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$; Let $a_n = \left(\frac{n}{10}\right)^n$, then $\lim_{n\to\infty} \left(\frac{n}{10}\right)^n = \infty$ so $\lim_{n\to\infty} (-1)^{n+1}\left(\frac{n}{10}\right)^n \neq 0$,

and series diverges by the nth-term test

4.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{10^n}{n^{10}}$; Let $a_n = \frac{10^n}{n^{10}}$, then

compare a_n and a_{n+1} :

$$\frac{10^n}{n^{10}} \sim \frac{10^{n+1}}{(n+1)^{10}} \quad \text{iff}$$

$$\frac{10^n}{10^{n+1}} \sim \frac{n^{10}}{(n+1)^{10}} \quad \text{iff} \quad \frac{1}{10} \sim \left(\frac{n}{n+1}\right)^{10}$$

$$\text{iff} \quad \left(\frac{1}{10}\right)^{1/10} \sim \frac{n}{n+1} \quad \text{iff} \quad (\approx 0.79) \sim \frac{n}{n+1} \; ;$$

we see that $0.79 < \frac{n}{n+1}$ for $n = 4, 5, 6, \ldots$; this means $a_n < a_{n+1}$ for $n = 4, 5, 6, \ldots$; since $a_4 > 0$ and the sequence is $\uparrow$, it follows that $\lim_{n\to\infty} \frac{10^n}{n^{10}} \neq 0$ and

hence $\lim_{n\to\infty} (-1)^{n+1} \cdot \frac{10^n}{n^{10}} \neq 0$, so series diverges by nth-term test

5.) $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1}{\ln n}$; Let $a_n = \frac{1}{\ln n}$, then a_n is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, so series converges by alternating series test

6.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\ln n}{n}$; Let $a_n = \frac{\ln n}{n}$, then a_n is $+$ (for $n \geq 2$) and $\lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$; $f(x) = \frac{\ln x}{x} \xrightarrow{D} f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2}$

$= \frac{1 - \ln x}{x^2}$ NO $+$ 0 $-$ f'

$x = 0$ $x = e \approx 2.7$

so a_n is $+$ for $n \geq 3$, $\downarrow$ for $n \geq 3$, and $\lim_{n\to\infty} a_n = 0$, so $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$ converges by alternating series test, so $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\ln n}{n}$ converges

9.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\sqrt{n}+1}{n+1}$; Let $a_n = \frac{\sqrt{n}+1}{n+1}$, then a_n is $+$ and $\lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{1} = 0$;

$f(x) = \frac{\sqrt{x}+1}{x+1} \xrightarrow{D} f'(x) = \frac{(x+1)\cdot\frac{1}{2\sqrt{x}} - (\sqrt{x}+1)(1)}{(x+1)^2}$

$$= \frac{\frac{x+1}{2\sqrt{x}} - \frac{2\sqrt{x}\,(\sqrt{x}+1)}{2\sqrt{x}}}{\frac{(x+1)^2}{1}} = \frac{(x+1)-(2x+2\sqrt{x})}{2\sqrt{x}\cdot(x+1)^2}$$

$$= \frac{1-x-2\sqrt{x}}{2\sqrt{x}\,(x+1)^2} \qquad \overset{-\qquad\qquad -}{\underset{x=1}{\longrightarrow}} \quad f'$$

so a_n is $+$, $\downarrow$, and $\lim\limits_{n\to\infty} a_n = 0$ so series converges by alternating series test

10.) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \cdot 3\,\frac{\sqrt{n+1}}{\sqrt{n}+1}$; Let $a_n = \frac{3\sqrt{n+1}}{\sqrt{n}+1}$,

then $\lim\limits_{n\to\infty} \frac{3\sqrt{n+1}}{\sqrt{n}+1} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{n\to\infty} \frac{\frac{3}{2\sqrt{n+1}}}{\frac{1}{2\sqrt{n}}}$

$= \lim\limits_{n\to\infty} 3\,\frac{\sqrt{n}}{\sqrt{n+1}} = \lim\limits_{n\to\infty} 3\sqrt{\frac{n}{n+1}} = \lim\limits_{n\to\infty} 3\sqrt{\frac{1}{1+\frac{1}{n}}}$

$= 3\sqrt{\frac{1}{1+0}} = 3(1) = 3$; thus,

$\lim\limits_{n\to\infty} (-1)^{n+1}\,\frac{3\sqrt{n+1}}{\sqrt{n}+1} \neq 0$ so series diverges by nth-term test

12.) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(0.1)^n}{n}$; consider series

$\sum\limits_{n=1}^{\infty} \frac{(0.1)^n}{n}$, then $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{(0.1)^{n+1}}{n+1}}{\frac{(0.1)^n}{n}}$

$= \lim_{n\to\infty} \frac{(0.1)^{n+1}}{(0.1)^n} \cdot \frac{n}{n+1} = \lim_{n\to\infty} (0.1) \cdot \frac{1}{1+\frac{1}{n}}$

$= (0.1) \cdot \frac{1}{1+0} = \frac{1}{10} < 1$, so series $\sum_{n=1}^{\infty} \frac{(0.1)^n}{n}$ converges by ratio test, and series $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{(0.1)^n}{n}$ converges absolutely

13.) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$ converges by the alternating series test since $a_n = \frac{1}{\sqrt{n}}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$; but the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series test ($p = \frac{1}{2} \le 1$), so series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$ converges conditionally

15.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n^3+1}$; consider series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$, then $\lim_{n\to\infty} \frac{\frac{n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^3+1}$

$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}} = \frac{1}{1+0} = 1$; since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test ($p=2>1$) then $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges by limit comparison test; so $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n^3+1}$ converges absolutely

16.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n!}{2^n}$; Let $a_n = \frac{n!}{2^n}$, then compare a_n and a_{n+1} :

$\frac{n!}{2^n} \sim \frac{(n+1)!}{2^{n+1}}$ iff $\frac{2^{n+1}}{2^n} \sim \frac{(n+1)!}{n!}$

iff $2 \sim n+1$; but $2 \leq n+1$ for $n = 1, 2, 3, \ldots$, so $a_n \leq a_{n+1}$; since $a_1 = \frac{1}{2}$ and $a_n \leq a_{n+1}$, then $\lim_{n\to\infty} a_n \neq 0$, so $\lim_{n\to\infty} (-1)^{n+1} \cdot \frac{n!}{2^n} \neq 0$ and series diverges

18.) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{\sin n}{n^2}$; consider series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$; $0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$, so $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$

converges by comparison test ; thus $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{\sin n}{n^2}$ converges absolutely

21.) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1+n}{n^2}$ converges by the alternating series test since $a_n = \frac{1+n}{n^2} = \frac{1}{n^2} + \frac{1}{n}$ is $+$, $\downarrow$, and

$\lim_{n\to\infty} a_n = 0$; consider series $\sum_{n=1}^{\infty} \frac{1+n}{n^2}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (by subtle facts about series)

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent p-series ($p = 2 > 1$) and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent p-series ($p = 1 \leq 1$).

23.) $\sum_{n=1}^{\infty} (-1)^n \cdot n^2 \cdot \left(\frac{2}{3}\right)^n$; consider series $\sum_{n=1}^{\infty} n^2 \cdot \left(\frac{2}{3}\right)^n$; then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot \left(\frac{2}{3}\right)^{n+1}}{n^2 \cdot \left(\frac{2}{3}\right)^n}$

$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \cdot \left(\frac{2}{3}\right) = (1)^2 \left(\frac{2}{3}\right) = \frac{2}{3} < 1$,

so $\sum_{n=1}^{\infty} n^2 \cdot \left(\frac{2}{3}\right)^n$ converges by ratio test ; thus, $\sum_{n=1}^{\infty} (-1)^n \cdot n^2 \cdot \left(\frac{2}{3}\right)^n$ converges absolutely

26.) $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1}{n \ln n}$ converges by alternating series test since $a_n = \frac{1}{n \ln n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$;

consider series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$; let $f(x) = \frac{1}{x \ln x}$, then f is $+$, $\downarrow$, and continuous for $x \geq 2$; thus,

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{A\to\infty} \int_2^A \frac{1}{x\ln x}\,dx$$

$$= \lim_{A\to\infty} \ln|\ln x|\Big|_2^A = \lim_{A\to\infty} \left(\ln|\ln A| - \ln|\ln 2|\right)$$

$= \infty$, so $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges by integral test, and $\sum_{n=2}^{\infty} (-1)^{n+1} \cdot \frac{1}{n\ln n}$ converges conditionally

27.) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{n+1}$; $\lim_{n\to\infty} \frac{n}{n+1} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \frac{1}{1} = 1$

so $\lim_{n\to\infty} (-1)^n \cdot \frac{n}{n+1} \neq 0$ and series diverges by the nth-term test

29.) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{100^n}{n!}$; consider series $\sum_{n=1}^{\infty} \frac{100^n}{n!}$; then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{100^{n+1}}{(n+1)!}}{\frac{100^n}{n!}}$

$= \lim_{n\to\infty} \frac{100^{n+1}}{100^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} 100 \cdot \frac{1}{n+1} = 0 < 1$,

so $\sum_{n=1}^{\infty} \frac{100^n}{n!}$ converges by ratio test and $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$ is absolutely convergent

34.) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \frac{\cos \pi}{1} + \frac{\cos 2\pi}{2} + \frac{\cos 3\pi}{3} + \cdots$

$= \frac{-1}{1} + \frac{1}{2} + \frac{-1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$;

Let $a_n = \frac{1}{n}$, then a_n is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$, so $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ converges by alternating series test; but series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series ($p=1 \le 1$), so $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ converges conditionally.

38.) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(n!)^2 \cdot 3^n}{(2n+1)!}$; consider series $\sum_{n=1}^{\infty} \frac{(n!)^2 \cdot 3^n}{(2n+1)!}$, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2 \cdot 3^{n+1}}{(2(n+1)+1)!}}{\frac{(n!)^2 \cdot 3^n}{(2n+1)!}}$

$$= \lim_{n\to\infty} \frac{(n+1)!(n+1)! \cdot 3^{n+1} \cdot (2n+1)!}{n! \cdot n! \cdot 3^n \cdot (2n+3)!}$$

$$= \lim_{n\to\infty} (n+1)(n+1) \cdot 3 \cdot \frac{1}{(2n+3)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{3n^2+6n+3}{4n^2+10n+6} = \lim_{n\to\infty} \frac{3+\frac{6}{n}+\frac{3}{n^2}}{4+\frac{10}{n}+\frac{6}{n^2}}$$

$$= \frac{3+0+0}{4+0+0} = \frac{3}{4} < 1,$$

so $\sum_{n=1}^{\infty} \frac{(n!)^2 \cdot 3^n}{(2n+1)!}$ converges by ratio test and $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(n!)^2 \cdot 3^n}{(2n+1)!}$ converges absolutely

39.) $\sum_{n=1}^{\infty} (-1)^n \cdot (\sqrt{n+1} - \sqrt{n})$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot (\sqrt{n+1} - \sqrt{n}) \cdot \frac{(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n+1}+\sqrt{n}}$, which converges by the alternating series test since $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} a_n = 0$; consider series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$; then $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{\sqrt{n}}}$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$= \frac{1}{1+1} = \frac{1}{2}$; since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \le 1$), $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by limit comparison test, and $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1}-\sqrt{n})$ is conditionally convergent

46.) $$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{10^n} = \underbrace{\frac{1}{10} + \frac{-1}{10^2} + \frac{1}{10^3} + \frac{-1}{10^4}}_{S_4 = 0.0909} + \underset{\substack{\uparrow \\ \text{error } R_4 < 0.00001}}{\frac{1}{10^5}} + \frac{-1}{10^6} + \cdots$$

so $S_4 = 0.0909$ estimates (under-estimate) the exact value of $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{10^n}$

with error at most $R_4 < 0.00001$

50.) $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n!} = \underbrace{1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \cdots + (-1)^n \cdot \frac{1}{n!}}_{S_n} + (-1)^{n+1} \cdot \frac{1}{(n+1)!} + \cdots$

error $R_n < \frac{1}{(n+1)!}$

require that $\frac{1}{(n+1)!} < 5 \times 10^{-6} = 0.000005$;

by calculator: $\frac{1}{8!} \approx 2.5 \times 10^{-5}$,

$\frac{1}{9!} \approx 2.7 \times 10^{-6} < 5 \times 10^{-6}$;

so choose $n = 8$; then

$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n!} \approx S_8 = \cancel{1} - \cancel{\frac{1}{1!}} + \frac{2}{2!} - \frac{3}{3!} + \frac{4}{4!} - \frac{5}{5!} + \frac{6}{6!} - \frac{7}{7!} + \frac{8}{8!}$

$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!}$

$= 0.632142857143$ and absolute error is at most 0.000005.

54.) $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \cdots$ (I)

$S_1 = 1$,

$\rightarrow S_2 = 1 - \frac{1}{2} = \frac{1}{2}$,

$S_3 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} = 1$,

$\rightarrow S_4 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} = \frac{2}{3}$,

$S_5 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} = 1$,

$\rightarrow S_6 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} = \frac{3}{4}$, ...

$S_{2n} = \frac{n}{n+1}$; $S_{2n+1} = 1$;

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots \qquad (\mathrm{II})$$

$S_1 = \frac{1}{2}$,

$S_2 = \frac{1}{2}+\frac{1}{6}=\frac{2}{3}$,

$S_3 = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} = \frac{3}{4}$,

$S_4 = \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20} = 4/5$, ...

$S_n = \frac{n}{n+1}$;

for series (I), since $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$

and $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} 1 = 1$, so sequence of partial sums converges to 1, so the series (I) has sum 1 ;

for series (II), the sequence of partial sums S_n satisfies

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$, so series (II) has sum 1 .

58.) Both $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

and $\sum_{n=1}^{\infty} (-1)^{n}\cdot\frac{1}{\sqrt{n}} = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \cdots$

converge by the alternating series test, but

$$\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{1}{\sqrt{n}}\cdot(-1)^{n}\cdot\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}(-1)\cdot\frac{1}{n} = -\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)$$

diverges by the p-series test.