Section 11.7

1.) a.) $\sum_{n=0}^{\infty} x^n$; $\lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} = \lim_{n\to\infty} |x| = |x| < 1$

$\to \underline{-1 < x < 1}$; $\underline{\text{check } x=1}$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$
diverges by nth-term test since
$\lim_{n\to\infty} 1 = 1 \neq 0$; $\underline{\text{check } x=-1}$: $\sum_{n=0}^{\infty} (-1)^n$

diverges by nth-term test since
$\lim_{n\to\infty} (-1)^n \neq 0$; so interval of
convergence is $\boxed{-1 < x < 1}$.

4.) a.) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$; $\lim_{n\to\infty} \frac{|3x-2|^{n+1}}{n+1} \cdot \frac{n}{|3x-2|^n}$

$= \lim_{n\to\infty} |3x-2| \cdot \frac{n}{n+1} = |3x-2| \cdot (1)$

$= |3x-2| < 1 \longrightarrow -1 < 3x-2 < 1 \longrightarrow$
$1 < 3x < 3 \to \underline{1/3 < x < 1}$; $\underline{\text{check } x=1}$:
$\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series
test $(p = 1 \leq 1)$; $\underline{\text{check } x = 1/3}$:
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test since
$a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{n} = 0$;
so interval of convergence is
$\boxed{\frac{1}{3} \leq x < 1}$.

6.) a.) $\sum_{n=0}^{\infty} (2x)^n$; $\lim_{n\to\infty} \frac{|2x|^{n+1}}{|2x|^n}$

$= \lim_{n\to\infty} |2x| = |2x| < 1 \to -1 < 2x < 1 \to$

$\underline{\frac{-1}{2} < x < \frac{1}{2}}$; $\underline{\text{check } x = 1/2}$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$

diverges by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$; $\underline{\text{check } x = -1/2}$:

$\sum_{n=0}^{\infty} (-1)^n$ diverges by nth-term test since $\lim_{n\to\infty} (-1)^n \neq 0$; so interval of convergence is

$\boxed{\frac{-1}{2} < x < \frac{1}{2}}$.

7.) a.) $\sum_{n=0}^{\infty} \frac{n x^n}{n+2}$; $\lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{n+3} \cdot \frac{n+2}{n|x|^n}$

$= \lim_{n\to\infty} |x| \cdot \frac{n+1}{n+3} \cdot \frac{n+2}{n} = |x| \cdot (1) \cdot (1) = |x| < 1$

$\rightarrow \underline{-1 < x < 1}$; $\underline{\text{check } x = 1}$: $\sum_{n=0}^{\infty} \frac{n \cdot 1^n}{n+2}$

$= \sum_{n=0}^{\infty} \frac{n}{n+2}$ diverges by nth-term test since $\lim_{n\to\infty} \frac{n}{n+2} = 1 \neq 0$;

$\underline{\text{check } x = -1}$: $\sum_{n=0}^{\infty} \frac{n(-1)^n}{n+2}$ diverges

by the nth-term test since $\lim_{n\to\infty} \frac{n}{n+2} = 1$ and so $\lim_{n\to\infty} (-1)^n \cdot \frac{n}{n+2} \neq 0$;

so interval of convergence is

$\boxed{-1 < x < 1}$.

10.) a.) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$; $\lim_{n\to\infty} \frac{|x-1|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|x-1|^n}$

$= \lim_{n\to\infty} |x-1| \cdot \sqrt{\frac{n}{n+1}} = |x-1| \cdot \sqrt{1} = |x-1| < 1$

$\rightarrow -1 < x-1 < 1 \rightarrow \underline{0 < x < 2}$; check $x=0$:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test since $a_n = \frac{1}{\sqrt{n}}$ is $+$, $\downarrow$, and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; check $x=2$:

$\sum_{n=1}^{\infty} \frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series test ($p = \frac{1}{2} \leq 1$); so interval of convergence is

$\boxed{0 \leq x < 2}$

12.) a.) $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$; $\lim_{n\to\infty} \frac{3^{n+1}|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n |x|^n}$

$= \lim_{n\to\infty} 3 \cdot \frac{1}{n+1} |x| = 3 \cdot (0) \cdot |x| = 0 < 1$

for all x-values, so interval of convergence is $\boxed{-\infty < x < \infty}$.

17.) a.) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$; $\lim_{n\to\infty} \frac{(n+1)|x+3|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n|x+3|^n}$

$= \lim_{n\to\infty} \frac{1}{5} \cdot \left(1 + \frac{1}{n}\right)|x+3| = \frac{1}{5} \cdot (1) \cdot |x+3|$

$= \frac{1}{5}|x+3| < 1 \rightarrow |x+3| < 5 \rightarrow -5 < x+3 < 5$

$\rightarrow \underline{-8 < x < 2}$; check $x=2$:

$\sum_{n=0}^{\infty} n \cdot \frac{\cancel{5^n}}{\cancel{5^n}} = \sum_{n=0}^{\infty} n$ diverges by nth-term test since $\lim_{n\to\infty} n = \infty \neq 0$;

check $x=-8$: $\sum_{n=0}^{\infty} n \cdot \frac{(-5)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n \cdot n$

diverges by the nth-term test since $\lim_{n\to\infty} (-1)^n \cdot n \neq 0$; so interval of convergence is $\boxed{-8 < x < 2}$.

22.) a.) $\sum_{n=1}^{\infty} (\ln n) x^n$; $\lim_{n\to\infty} \frac{\ln(n+1) \cdot |x|^{n+1}}{\ln n \cdot |x|^n}$

$= \lim_{n\to\infty} |x| \cdot \frac{\ln n+1}{\ln n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} |x| \cdot \frac{\frac{1}{n+1}}{\frac{1}{n}}$

$= \lim_{n\to\infty} |x| \cdot \frac{n}{n+1} = |x| \cdot (1) = |x| < 1 \longrightarrow$

$\underline{-1 < x < 1}$; check $x=1$: $\sum_{n=1}^{\infty} (\ln n) \cdot 1^n$

$= \sum_{n=1}^{\infty} (\ln n)$ diverges by nth-term test since $\lim_{n\to\infty} \ln n = \infty \neq 0$;

check $x=-1$: $\sum_{n=1}^{\infty} (\ln n) \cdot (-1)^n$ diverges

by nth-term test since

$\lim_{n\to\infty} \ln n = \infty$, so $\lim_{n\to\infty} (-1)^n \cdot \ln n \neq 0$;

so interval of convergence is

$\boxed{-1 < x < 1}$.

23.) a.) $\sum_{n=1}^{\infty} n^n x^n$; $\lim_{n\to\infty} \frac{(n+1)^{n+1}\cdot|x|^{n+1}}{n^n\cdot|x|^n}$

$= \lim_{n\to\infty} (n+1)\cdot\frac{(n+1)^n}{n^n}\cdot|x|$

$= \lim_{n\to\infty} (n+1)\cdot\left(1+\frac{1}{n}\right)^n\cdot|x| = \begin{cases}\infty(1)\cdot|x| = \infty, \text{ if } x\neq 0\\ 0<1 \text{ if } x=0\end{cases}$;

so interval of convergence is $\boxed{x=0}$.

25.) a.) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+2)^n}{n2^n}$;

$\lim_{n\to\infty}\frac{|x+2|^{n+1}}{(n+1)\cdot 2^{n+1}}\cdot\frac{n\cdot 2^n}{|x+2|^n} = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{1}{2}\cdot|x+2|$

$= (1)\cdot\frac{1}{2}\cdot|x+2| = \frac{1}{2}|x+2| < 1 \longrightarrow$

$|x+2| < 2 \longrightarrow -2 < x+2 < 2 \longrightarrow$

$\underline{-4 < x < 0}$; check $\underline{x=0}$: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\cancel{2^n}}{n\cdot\cancel{2^n}}$

$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by the alternating series test since $a_n = \frac{1}{n}$ is $+$, $\downarrow$, and $\lim_{n\to\infty}\frac{1}{n} = 0$; check $\underline{x=-4}$:

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\cdot(-1)^n\cdot\frac{1}{n}$

$= \sum_{n=1}^{\infty}(-1)\cdot(-1)^n\cdot(-1)^n\cdot\frac{1}{n} = \sum_{n=1}^{\infty}(-1)(-1)^{2n}\cdot\frac{1}{n}$

$= \sum_{n=1}^{\infty}(-1)\,(\underbrace{(-1)^2}_{1})^n\cdot\frac{1}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series test $(p=1\le 1)$; so interval of convergence is $\boxed{-4<x\le 0}$.

27.) a.) $\sum_{n=2}^{\infty}\frac{x^n}{n(\ln n)^2}$; $\lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)(\ln(n+1))^2}\cdot\frac{n(\ln n)^2}{|x|^n}$

$=\lim_{n\to\infty}|x|\cdot\frac{n}{n+1}\cdot\left(\frac{\ln n}{\ln(n+1)}\right)^2$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=}\lim_{n\to\infty}|x|\cdot\frac{1}{1}\cdot\left(\frac{\frac{1}{n}}{\frac{1}{n+1}}\right)^2=\lim_{n\to\infty}|x|\cdot\left(\frac{n+1}{n}\right)^2$

$=|x|\cdot(1)^2=|x|<1\longrightarrow \underline{-1<x<1}$;

<u>check $x=1$</u> : $\sum_{n=2}^{\infty}\frac{1^n}{n(\ln n)^2}=\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$;

$f(x)=\frac{1}{x(\ln x)^2}$ is $+$, $\downarrow$, and continuous for $x\ge 2$ and

$\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx=\lim_{A\to\infty}\frac{-1}{\ln x}\Big|_2^A$

$=\lim_{x\to\infty}\left(\frac{-1}{\ln A}-\frac{-1}{\ln 2}\right)=\frac{1}{\ln 2}$ (converges)

so series converges by integral test;

<u>check $x=-1$</u> : $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\ln n)^2}$ converges

by the alternating series test since $a_n=\frac{1}{n(\ln n)^2}$ is $+$, $\downarrow$, and

$\lim_{n\to\infty}\frac{1}{n(\ln n)^2}=0$; so interval of convergence is $\boxed{-1\le x\le 1}$.

34.) $\sum_{n=0}^{\infty}\frac{(x+1)^{2n}}{9^n}$; $\lim_{n\to\infty}\frac{|x+1|^{2(n+1)}}{9^{n+1}}\cdot\frac{9^n}{|x+1|^{2n}}$

$= \lim_{n\to\infty} \frac{1}{9}|x+1|^2 = \frac{1}{9}|x+1|^2 < 1 \longrightarrow$

$|x+1|^2 < 9 \longrightarrow |x+1| < 3 \longrightarrow$

$-3 < x+1 < 3 \longrightarrow \underline{-4 < x < 2}$;

<u>check $x=2$</u>: $\sum_{n=0}^{\infty} \frac{3^{2n}}{9^n} = \sum_{n=0}^{\infty} \frac{9^n}{9^n}$

$= \sum_{n=0}^{\infty} 1$ diverges by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$; <u>check $x=-4$</u>:

$\sum_{n=0}^{\infty} \frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty} \frac{9^n}{9^n} = \sum_{n=0}^{\infty} 1$ diverges

by nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$;
so interval of convergence is
$\boxed{-4 < x < 2}$;

$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty} \left(\frac{(x+1)^2}{9}\right)^n$

$= 1 + \left(\frac{(x+1)^2}{9}\right) + \left(\frac{(x+1)^2}{9}\right)^2 + \left(\frac{(x+1)^2}{9}\right)^3 + \cdots$

$= \frac{1}{1 - \frac{(x+1)^2}{9}} \cdot \frac{9}{9} = \frac{9}{9-(x^2+2x+1)} = \frac{9}{8-x^2-2x}$.

36.) $\sum_{n=0}^{\infty} (\ln x)^n$; $\lim_{n\to\infty} \frac{|\ln x|^{n+1}}{|\ln x|^n}$

$= \lim_{n\to\infty} |\ln x| = |\ln x| < 1 \longrightarrow$

$-1 < \ln x < 1 \longrightarrow e^{-1} < x < e^{1} \longrightarrow$

$\underline{\frac{1}{e} < x < e}$; $\underline{\text{check } x = e}$: $\sum_{n=0}^{\infty} (\ln e)^n$

$= \sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$ diverges by

nth-term test since $\lim_{n\to\infty} 1 = 1 \neq 0$;

$\underline{\text{check } x = \frac{1}{e}}$: $\sum_{n=0}^{\infty} (\ln e^{-1})^n = \sum_{n=0}^{\infty} (-1)^n$

diverges by nth-term test since

$\lim_{n\to\infty} (-1)^n \neq 0$; so interval of

convergence is $\boxed{\frac{1}{e} < x < e}$;

$\sum_{n=0}^{\infty} (\ln x)^n = 1 + (\ln x) + (\ln x)^2 + (\ln x)^3 + \cdots$

$= \dfrac{1}{1 - \ln x}$.

46.) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$; begin with

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \xrightarrow{D}$

$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \cdots \longrightarrow$

$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots \xrightarrow{D}$

$\frac{(1-x)^2 \cdot (1) - x \cdot 2(1-x)(-1)}{(1-x)^4} = 1 + 4x + 9x^2 + 16x^3 + \cdots$

$\longrightarrow$

$$\frac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + \cdots \longrightarrow$$

Let $x = \frac{1}{2}$

$$\frac{x(1+x)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + \cdots \longrightarrow$$

$$\frac{\frac{1}{2}\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^3} = \frac{1}{2} + \frac{4}{2^2} + \frac{9}{2^3} + \frac{16}{2^4} + \cdots$$

$$= \frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots = \sum_{n=0}^{\infty} \frac{n^2}{2^n} \longrightarrow$$

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} = \frac{3/4}{1/8} = \frac{3}{4} \cdot \frac{8}{1} = 6 \, .$$

Section 11.8

Taylor polynomial of degree n:

$P_n(x;a) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n$, where

$$a_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n = 0, 1, 2, 3, \ldots$$

1.) $f(x) = \ln x$, $a = 1$:

$f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, so

$a_0 = \frac{f(1)}{0!} = \frac{\ln 1}{1} = \frac{0}{1} = 0$,

$a_1 = \frac{f'(1)}{1!} = \frac{1}{1} = 1$, $a_2 = \frac{f''(1)}{2!} = \frac{-1}{2}$,

$a_3 = \frac{f'''(1)}{3!} = \frac{2}{6} = \frac{1}{3}$; then

$P_0(x;1) = a_0 = 0$,

$P_1(x;1) = a_0 + a_1(x-1) = 1\cdot(x-1) = x-1$,

$P_2(x;1) = a_0 + a_1(x-1) + a_2(x-1)^2$

$= (x-1) + \frac{-1}{2}(x-1)^2$,

$P_3(x;1) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$

$= (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

4.) $f(x) = \frac{1}{x+2} = (x+2)^{-1}$, $a = 0$:

$f'(x) = -(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$,

$f'''(x) = -6(x+2)^{-4}$, so

$a_0 = \frac{f(0)}{0!} = \frac{1/2}{1} = 1/2$, $a_1 = \frac{f'(0)}{1!} = \frac{-1/4}{1} = -1/4$,

$$a_2 = \frac{f''(0)}{2!} = \frac{2 \cdot 1/8}{2} = \frac{1}{8},$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{-6 \cdot 1/16}{6} = \frac{-1}{16}; \text{ then}$$

$$P_0(x;0) = a_0 = 1/2,$$

$$P_1(x;0) = a_0 + a_1 x = \frac{1}{2} - \frac{1}{4}x,$$

$$P_2(x;0) = a_0 + a_1 x + a_2 x^2$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2,$$

$$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3$$

6.) $f(x) = \cos x$, $a = \frac{\pi}{4}$:

$f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$,

so $a_0 = \frac{f(\frac{\pi}{4})}{0!} = \frac{\cos\frac{\pi}{4}}{1} = \frac{\sqrt{2}}{2}$,

$$a_1 = \frac{f'(\frac{\pi}{4})}{1!} = \frac{-\sin\frac{\pi}{4}}{1} = \frac{-\sqrt{2}}{2},$$

$$a_2 = \frac{f''(\frac{\pi}{4})}{2!} = \frac{-\cos\frac{\pi}{4}}{2} = \frac{-\frac{\sqrt{2}}{2}}{2} = \frac{-\sqrt{2}}{4},$$

$$a_3 = \frac{f'''(\frac{\pi}{4})}{3!} = \frac{\sin\frac{\pi}{4}}{6} = \frac{\frac{\sqrt{2}}{2}}{6} = \frac{\sqrt{2}}{12}; \text{ then}$$

$$P_0(x;\tfrac{\pi}{4}) = a_0 = \frac{\sqrt{2}}{2},$$

$$P_1(x;\tfrac{\pi}{4}) = a_0 + a_1(x - \tfrac{\pi}{4}) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \tfrac{\pi}{4}),$$

$$P_2(x;\tfrac{\pi}{4}) = a_0 + a_1(x - \tfrac{\pi}{4}) + a_2(x - \tfrac{\pi}{4})^2$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 ,$$

$$P_3\left(x;\frac{\pi}{4}\right) = a_0 + a_1\left(x-\frac{\pi}{4}\right) + a_2\left(x-\frac{\pi}{4}\right)^2 + a_3\left(x-\frac{\pi}{4}\right)^3$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3$$

8.) $f(x) = \sqrt{x+4}$, $a = 0$:

$f'(x) = \frac{1}{2}(x+4)^{-1/2}$, $f''(x) = \frac{-1}{4}(x+4)^{-3/2}$,

$f'''(x) = \frac{3}{8}(x+4)^{-5/2}$; so

$$a_0 = \frac{f(0)}{0!} = \frac{\sqrt{4}}{1} = 2 , \quad a_1 = \frac{f'(0)}{1!} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{1} = \frac{1}{4} ,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{\frac{-1}{4}\cdot\frac{1}{8}}{2} = \frac{-1}{64} ,$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{3/8 \cdot 1/32}{6} = \frac{1}{512} \text{ ; then}$$

$P_0(x;0) = a_0 = 2$,

$P_1(x;0) = a_0 + a_1 x = 2 + \frac{1}{4}x$,

$P_2(x;0) = a_0 + a_1 x + a_2 x^2 = 2 + \frac{1}{4}x - \frac{1}{64}x^2$,

$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3$$

9.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

10.) $e^{\frac{x}{2}} = 1 + \left(\frac{x}{2}\right) + \frac{\left(\frac{x}{2}\right)^2}{2!} + \frac{\left(\frac{x}{2}\right)^3}{3!} + \cdots$

$= 1 + \frac{x}{2} + \frac{x^2}{2^2 2!} + \frac{x^3}{2^3 \cdot 3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{2^n \cdot n!}$

11.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$, so

$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$

$= 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$

13.) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$,

so $\sin 3x = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \cdots$

$= 3x - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7 + \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3^{2n+1}}{(2n+1)!} x^{2n+1}$

20) $f(x) = (x+1)^2$, $a = 0$:

$f'(x) = 2(x+1)$, $f''(x) = 2$, $f'''(x) = 0$, ...

so $a_0 = \frac{f(0)}{0!} = \frac{1^2}{1} = 1$, $a_1 = \frac{f'(0)}{1!} = \frac{2}{1} = 2$,

$a_2 = \frac{f''(0)}{2!} = \frac{2}{2} = 1$, $a_3 = \frac{f'''(0)}{3!} = \frac{0}{6} = 0$,

$a_4 = a_5 = a_6 = \cdots = 0$; then

$(x+1)^2 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$= 1 + 2x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \cdots$

$= 1 + 2x + x^2$

23.) $f(x) = x^4 + x^2 + 1$, $a = -2$:

$f'(x) = 4x^3 + 2x$, $f''(x) = 12x^2 + 2$,

$f'''(x) = 24x$, $f^{(4)}(x) = 24$, $f^{(5)}(x) = 0$,

$f^{(6)}(x) = f^{(7)}(x) = f^{(8)}(x) = \cdots = 0$; so

$a_0 = \frac{f(-2)}{0!} = \frac{21}{1} = 21$, $a_1 = \frac{f'(-2)}{1!} = \frac{-36}{1} = -36$,

$a_2 = \frac{f''(-2)}{2!} = \frac{50}{2} = 25$, $a_3 = \frac{f'''(-2)}{3!} = \frac{-48}{6} = -8$,

$a_4 = \frac{f^{(4)}(-2)}{4!} = \frac{24}{24} = 1$, $a_5 = a_6 = a_7 = \cdots = 0$;

then $x^4 + x^2 + 1 = a_0 + a_1(x-(-2)) + a_2(x-(-2))^2 + \cdots$

$= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4 + 0(x+2)^5 + \cdots$

$= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$

25.) $f(x) = \frac{1}{x^2}$, $a = 1$:

$f'(x) = -2x^{-3}$, $f''(x) = 3 \cdot 2x^{-4} = 3!\,x^{-4}$,

$f'''(x) = -4 \cdot 3!\,x^{-5} = -4!\,x^{-5}$,

$f^{(4)}(x) = 5 \cdot 4!\,x^{-6} = 5!\,x^{-6}$, $\ldots$,

$f^{(n)}(x) = (n+1)!\,x^{-(n+2)}$ for $n = 0, 1, 2, 3, \ldots$;

then

$a_0 = \frac{f(1)}{0!} = \frac{1}{1} = 1$, $a_1 = \frac{f'(1)}{1!} = \frac{-2}{1} = -2$,

$a_2 = \frac{f''(1)}{2!} = \frac{3!}{2!} = 3$, $a_3 = \frac{f'''(1)}{3!} = \frac{-4!}{3!} = -4$,

$a_4 = \frac{f^{(4)}(1)}{4!} = \frac{5!}{4!} = 5$, $\ldots$, $a_n = (-1)^n \cdot (n+1)$

for $n = 0, 1, 2, 3, \ldots$; then

$$\frac{1}{x^2} = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots$$
$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \cdot (n+1) \cdot (x-1)^n .$$

26.) $\frac{x}{1-x} = x \cdot \frac{1}{1-x} = x \cdot (1 + x + x^2 + x^3 + \cdots)$
$= x + x^2 + x^3 + x^4 + \cdots = \sum_{n=1}^{\infty} x^n$

27.) $f(x) = e^x$, $a = 2$:
$f'(x) = e^x$, $f^{(n)}(x) = e^x$ for $n = 0, 1, 2, 3, \ldots$;
so $a_0 = \frac{f(2)}{0!} = \frac{e^2}{1} = e^2$, $a_1 = \frac{f'(2)}{1!} = \frac{e^2}{1} = e^2$,
$a_2 = \frac{f''(2)}{2!} = \frac{e^2}{2!}$, $a_3 = \frac{e^2}{3!}$, $\ldots$, $a_n = \frac{e^2}{n!}$ for
$n = 0, 1, 2, 3, \ldots$; then
$$e^x = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + \cdots$$
$$= e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n .$$

33.) $f(x) = \ln(\cos x)$, $a = 0$:
$f'(x) = \frac{1}{\cos x} \cdot -\sin x = -\tan x$,
$f''(x) = -\sec^2 x$; so $a_0 = \frac{f(0)}{0!} = \frac{\ln(\cos 0)}{1}$
$= \frac{\ln 1}{1} = \frac{0}{1} = 0$, $a_1 = \frac{f'(0)}{1!} = \frac{-\tan 0}{1} = 0$,

$$a_2 = \frac{f''(0)}{2!} = \frac{-\sec^2 0}{2} = \frac{-1}{2} \; ; \text{ then}$$

a.) $P_1(x;0) = a_0 + a_1 x = 0 + 0 \cdot x = 0$

b.) $P_2(x;0) = a_0 + a_1 x + a_2 x^2$

$= 0 + 0 \cdot x - \frac{1}{2} x^2 = \frac{-1}{2} x^2$

35.) $f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}, \quad a = 0:$

$$f'(x) = \frac{-1}{2}(1-x^2)^{-3/2} \cdot (-2x) = \frac{x}{(1-x^2)^{3/2}},$$

$$f''(x) = \frac{(1-x^2)^{3/2}(1) - x \cdot \frac{3}{2}(1-x^2)^{1/2} \cdot (-2x)}{(1-x^2)^3}$$

$$= \frac{(1-x^2)^{1/2} \cdot [(1-x^2) + 3x^2]}{(1-x^2)^3} = \frac{1-2x^2}{(1-x^2)^{5/2}};$$

then $a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1,$

$a_1 = \frac{f'(0)}{1!} = \frac{0}{1} = 0, \quad a_2 = \frac{f''(0)}{2!} = \frac{1}{2} \; ; \text{ then}$

a.) $P_1(x;0) = a_0 + a_1 x = 1 + 0 \cdot x = 1$

b.) $P_2(x;0) = a_0 + a_1 x + a_2 x^2$

$= 1 + 0 \cdot x + \frac{1}{2} x^2 = 1 + \frac{1}{2} x^2$