Section 12.3

1.) $\vec{v} = \overrightarrow{\langle 2, -4, \sqrt{5} \rangle}$, $\vec{u} = \overrightarrow{\langle -2, 4, -\sqrt{5} \rangle}$

a.) $\vec{v} \cdot \vec{u} = (-4) + (-16) + (-5) = -25$,

$|\vec{u}| = \sqrt{4+16+5} = \sqrt{25} = 5$,

$|\vec{v}| = \sqrt{4+16+5} = \sqrt{25} = 5$.

b.) $\cos\theta = \dfrac{\vec{v} \cdot \vec{u}}{|\vec{v}||\vec{u}|} = \dfrac{-25}{(5)(5)} = -1 \ (\rightarrow \theta = \pi)$

c.) $|\vec{u}|\cos\theta = (5)(-1) = -5$

d.) $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} = \dfrac{-25}{(5)^2}\vec{v}$

$= -\vec{v} = \overrightarrow{\langle -2, 4, -\sqrt{5} \rangle}$

4.) $\vec{v} = \overrightarrow{\langle 2, 10, -11 \rangle}$, $\vec{u} = \overrightarrow{\langle 2, 2, 1 \rangle}$

a.) $\vec{v} \cdot \vec{u} = 4 + 20 + (-11) = 13$,

$|\vec{u}| = \sqrt{4+4+1} = \sqrt{9} = 3$,

$|\vec{v}| = \sqrt{4+100+121} = \sqrt{225} = 15$

b.) $\cos\theta = \dfrac{\vec{v} \cdot \vec{u}}{|\vec{v}||\vec{u}|} = \dfrac{13}{(15)(3)} = \dfrac{13}{45}$

c.) $|\vec{u}|\cos\theta = (3) \cdot \left(\dfrac{13}{45}\right) = \dfrac{13}{15}$

d.) $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} = \dfrac{13}{(15)^2}\vec{v}$

$= \dfrac{13}{225}\overrightarrow{\langle 2, 10, -11 \rangle} = \overrightarrow{\left\langle \dfrac{26}{225}, \dfrac{130}{225}, \dfrac{-143}{225} \right\rangle}$

7.) $\vec{v} = \overrightarrow{(5,1)}$, $\vec{u} = \overrightarrow{(2, \sqrt{17})}$

a.) $\vec{v} \cdot \vec{u} = 10 + \sqrt{17}$,

$|\vec{u}| = \sqrt{4+17} = \sqrt{21}$,

$|\vec{v}| = \sqrt{25+1} = \sqrt{26}$

b.) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{10+\sqrt{17}}{\sqrt{21}\sqrt{26}} = \dfrac{10+\sqrt{17}}{\sqrt{546}}$

c.) $|\vec{u}|\cos\theta = \cancel{\sqrt{21}} \cdot \dfrac{10+\sqrt{17}}{\cancel{\sqrt{21}}\sqrt{26}} = \dfrac{10+\sqrt{17}}{\sqrt{26}}$

d.) $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$

$= \dfrac{10+\sqrt{17}}{26}\vec{v} = \overrightarrow{\left(\dfrac{5(10+\sqrt{17})}{26}, \dfrac{10+\sqrt{17}}{26}\right)}$

10.) $\vec{u} = \overrightarrow{(2,-2,1)}$, $\vec{v} = \overrightarrow{(3,0,4)}$, so

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{6+0+4}{\sqrt{9}\cdot\sqrt{25}} = \dfrac{10}{(3)(5)} = \dfrac{2}{3} \rightarrow$

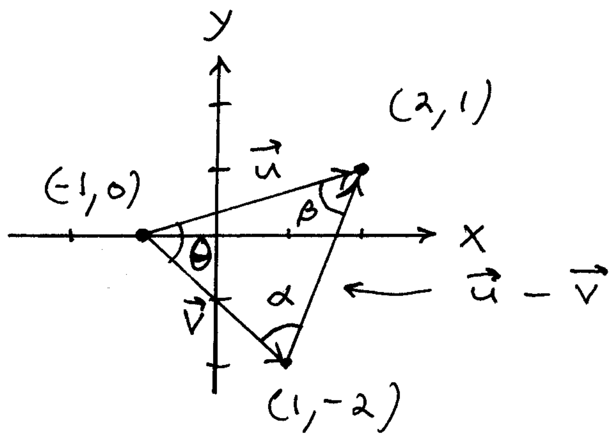
$\theta = \arccos\frac{2}{3} \approx 0.84$ radians

12.) $\vec{u} = \overrightarrow{(1, \sqrt{2}, -\sqrt{2})}$, $\vec{v} = \overrightarrow{(-1,1,1)}$, so

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{-1+\sqrt{2}-\sqrt{2}}{\sqrt{5}\sqrt{3}} = \dfrac{-1}{\sqrt{15}} \rightarrow$

$\theta = \arccos\left(\frac{-1}{\sqrt{15}}\right) \approx 1.83$ radians

13.)



$\vec{u} = \overrightarrow{(3,1)}$, $\vec{v} = \overrightarrow{(2,-2)}$, $\vec{u}-\vec{v} = \overrightarrow{(1,3)}$;

angle between $\vec{u}$ and $\vec{v}$ satisfies

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{6-2}{\sqrt{10}\cdot\sqrt{8}} = \frac{4}{\sqrt{80}} = \frac{1}{\sqrt{5}} \longrightarrow$$

$$\theta = \arccos\frac{1}{\sqrt{5}} \approx 63.4^\circ \;;$$

angle between $-\vec{v}$ and $\vec{u}-\vec{v}$ satisfies

$$\cos\alpha = \frac{-\vec{v}\cdot(\vec{u}-\vec{v})}{|-\vec{v}||\vec{u}-\vec{v}|} = \frac{\overrightarrow{(-2,2)}\cdot\overrightarrow{(1,3)}}{|\overrightarrow{(-2,2)}||\overrightarrow{(1,3)}|}$$

$$= \frac{-2+6}{\sqrt{8}\sqrt{10}} = \frac{4}{\sqrt{80}} = \frac{1}{\sqrt{5}} \longrightarrow$$

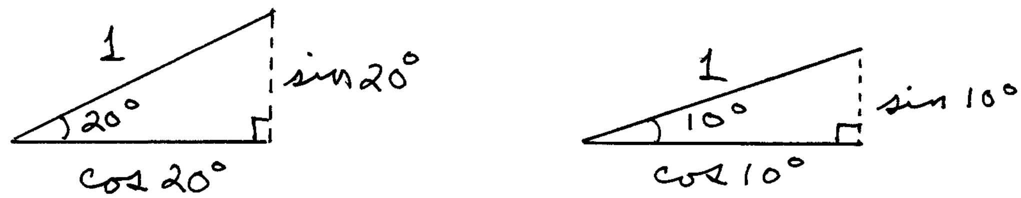
$$\alpha = \arccos\frac{1}{\sqrt{5}} \approx 63.4^\circ \;;$$

$$\theta+\alpha+\beta = 180^\circ \longrightarrow$$

$$\beta = 180^\circ - \alpha - \theta \approx 53.2^\circ$$

16.)

assume $\vec{u}$ and $\vec{v}$ are unit vectors in space. Then



so that

$$\vec{u} = 0\,\vec{i} + (\cos 20°)\,\vec{j} + (\sin 20°)\,\vec{k}$$

and $$\vec{v} = (\cos 10°)\,\vec{i} + 0 \cdot \vec{j} + (\sin 10°)\,\vec{k} \quad ;$$

the angle θ in the water main is the angle between

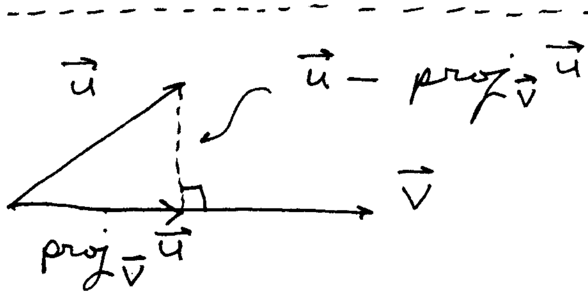
$-\vec{u}$ and $\vec{v}$

(since tails must be together !!)

so

$$\cos\theta = \frac{(-\vec{u})\cdot\vec{v}}{|-\vec{u}|\,|\vec{v}|} = \frac{0 + 0 - (\sin 20°)(\sin 10°)}{(1)\quad(1)} \longrightarrow$$

$$\theta = \arccos(-\sin 20° \cdot \sin 10°) \approx 93.4°$$



18.) $\vec{u} = \vec{j} + \vec{k}$, $\vec{v} = \vec{i} + \vec{j}$ then

$\vec{u} = \overrightarrow{(0,1,1)}$, $\vec{v} = \overrightarrow{(1,1,0)}$ so

$\vec{u} \cdot \vec{v} = 0 + 1 + 0 = 1$, $|\vec{u}| = \sqrt{2}$, $|\vec{v}| = \sqrt{2}$;

$$proj_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right) \vec{v} = \frac{1}{2} \vec{v} = \overrightarrow{(\tfrac{1}{2}, \tfrac{1}{2}, 0)} \quad \text{and}$$

$$\vec{u} - proj_{\vec{v}} \vec{u} = \overrightarrow{(0,1,1)} - \overrightarrow{(\tfrac{1}{2}, \tfrac{1}{2}, 0)} = \overrightarrow{(-\tfrac{1}{2}, \tfrac{1}{2}, 1)} ;$$

so $$\vec{u} = \underbrace{\overrightarrow{(\tfrac{1}{2}, \tfrac{1}{2}, 0)}}_{\parallel \text{ to } \vec{v}} + \underbrace{\overrightarrow{(\tfrac{-1}{2}, \tfrac{1}{2}, 1)}}_{\perp \text{ to } \vec{v}} .$$

19.) $\vec{u} = \overrightarrow{(8,4,-12)}$, $\vec{v} = \overrightarrow{(1,2,-1)}$ so

$\vec{u} \cdot \vec{v} = 8 + 8 + 12 = 28$, $|\vec{u}| = \sqrt{224} = 4\sqrt{14}$,

$|\vec{v}| = \sqrt{6}$; $proj_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right) \vec{v}$

$$= \frac{28}{6} \vec{v} = \frac{14}{3} \vec{v} = \overrightarrow{\left(\frac{14}{3}, \frac{28}{3}, \frac{-14}{3}\right)} \quad \text{and}$$

$$\vec{u} - proj_{\vec{v}} \vec{u} = \overrightarrow{(8,4,-12)} - \overrightarrow{\left(\frac{14}{3}, \frac{28}{3}, \frac{-14}{3}\right)}$$

$$= \overrightarrow{\left(\frac{10}{3}, \frac{-16}{3}, \frac{-22}{3}\right)} \quad ; \text{ so}$$

$$\vec{u} = \underbrace{\overrightarrow{\left(\frac{14}{3}, \frac{28}{3}, \frac{-14}{3}\right)}}_{\parallel \text{ to } \vec{v}} + \underbrace{\overrightarrow{\left(\frac{10}{3}, \frac{-16}{3}, \frac{-22}{3}\right)}}_{\perp \text{ to } \vec{v}}$$

21.) Vectors $\vec{v}_1 + \vec{v}_2$ and $\vec{v}_1 - \vec{v}_2$ are $\perp$

iff $(\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 - \vec{v}_2) = 0$

iff $(\vec{v}_1 + \vec{v}_2) \cdot \vec{v}_1 - (\vec{v}_1 + \vec{v}_2) \cdot \vec{v}_2 = 0$

(by property 3)

iff $\vec{v}_1 \cdot \vec{v}_1 + \vec{v}_2 \cdot \vec{v}_1 - \vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2 \cdot \vec{v}_2 = 0$

(by property 3)

iff $|\vec{v}_1|^2 + \cancel{\vec{v}_1 \cdot \vec{v}_2} - \cancel{\vec{v}_1 \cdot \vec{v}_2} - |\vec{v}_2|^2 = 0$

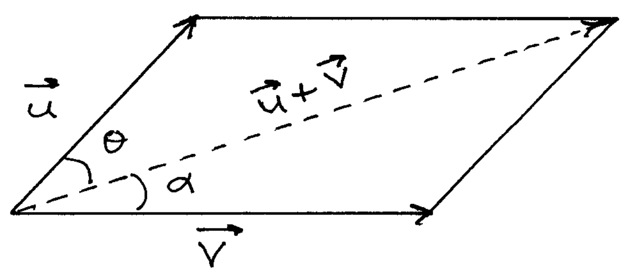
(by properties 1 and 4)

iff $|\vec{v}_1|^2 = |\vec{v}_2|^2$

iff $|\vec{v}_1| = |\vec{v}_2|$, i.e.,

vectors $\vec{v}_1$ and $\vec{v}_2$ have the same magnitude.

26.) Show ~~that~~ angles $\theta = \alpha$ if $|\vec{u}| = |\vec{v}|$:



$$\cos\theta = \frac{\vec{u} \cdot (\vec{u}+\vec{v})}{|\vec{u}|\,|\vec{u}+\vec{v}|} = \frac{\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{u}+\vec{v}|}$$

$$= \frac{|\vec{u}|^2 + \vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{u}+\vec{v}|} \quad ;$$

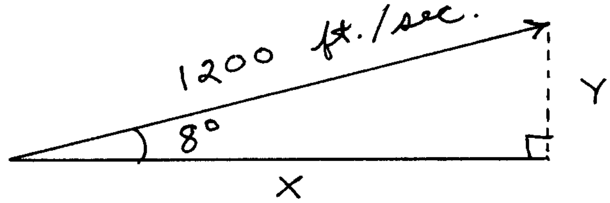
$$\cos\alpha = \frac{\vec{v} \cdot (\vec{u}+\vec{v})}{|\vec{v}|\,|\vec{u}+\vec{v}|} = \frac{\vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}}{|\vec{v}|\,|\vec{u}+\vec{v}|}$$

$$= \frac{\vec{u} \cdot \vec{v} + |\vec{v}|^2}{|\vec{v}|\,|\vec{u}+\vec{v}|} = \frac{\vec{u} \cdot \vec{v} + |\vec{u}|^2}{|\vec{u}|\,|\vec{u}+\vec{v}|}$$

(since $|\vec{u}| = |\vec{v}|$) ;

then $\cos\theta = \cos\alpha \longrightarrow \theta = \alpha$.

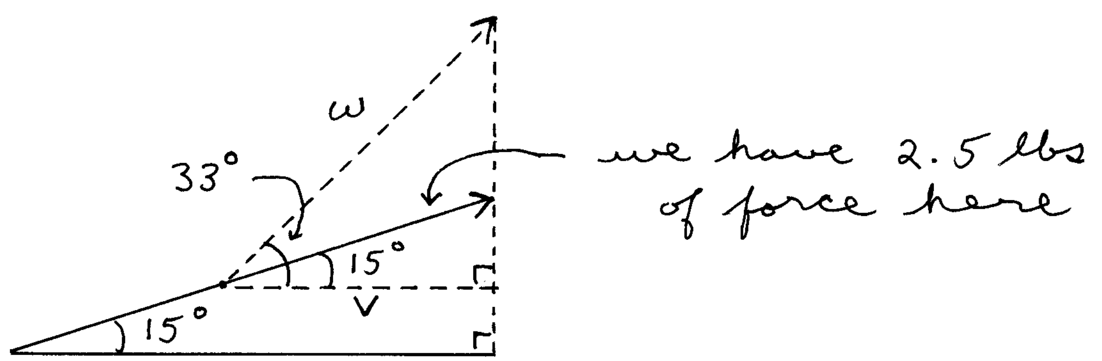
27.)



$\cos 8° = \dfrac{X}{1200} \longrightarrow X = 1200\cos 8°$
≈ 1188.32 ft./sec. ;

$\sin 8° = \dfrac{Y}{1200} \longrightarrow Y = 1200 \sin 8°$
≈ 167.01 ft./sec.

28.)



$\cos 15° = \dfrac{V}{2.5} \longrightarrow V = 2.5\cos 15°$;

$\cos 33° = \dfrac{V}{w} \longrightarrow$

$w = \dfrac{V}{\cos 33°} = \dfrac{2.5\cos 15°}{\cos 33°} \approx 2.88$ lbs.

29.) a.) $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta \longrightarrow$

$|\vec{u}\cdot\vec{v}| = \big|\, |\vec{u}||\vec{v}|\cos\theta \,\big|$

$= |\vec{u}||\vec{v}|\cdot|\cos\theta|$

$\leq |\vec{u}||\vec{v}|\cdot(1) \longrightarrow$

$|\vec{u}\cdot\vec{v}| \leq |\vec{u}||\vec{v}|$.

6.) If $|\cos\theta| = 1$, then

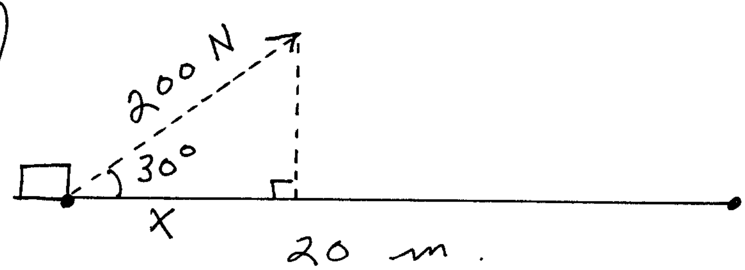
$$|\vec{u}\cdot\vec{v}| = |\vec{u}||\vec{v}|\cdot|\cos\theta|$$
$$= |\vec{u}||\vec{v}|\cdot(1)$$
$$= |\vec{u}||\vec{v}|.$$

This equality holds if $\theta = 0°$ (vectors have same direction) or if $\theta = 180°$ (vectors have opposite direction)

31.) Assume $\vec{u}_1$ is $\perp$ to $\vec{u}_2$ and $|\vec{u}_1| = 1 = |\vec{u}_2|$. If $\vec{v} = a\vec{u}_1 + b\vec{u}_2$, then

$$\vec{v}\cdot\vec{u}_1 = (a\vec{u}_1 + b\vec{u}_2)\cdot\vec{u}_1$$
$$= a\cdot(\vec{u}_1\cdot\vec{u}_1) + b\cdot(\vec{u}_2\cdot\vec{u}_1)$$
$$= a\,|\vec{u}_1|^2 + b\cdot(0) \quad (\text{since } \vec{u}_2 \perp \vec{u}_1)$$
$$= a(1)^2 = a$$

45.)

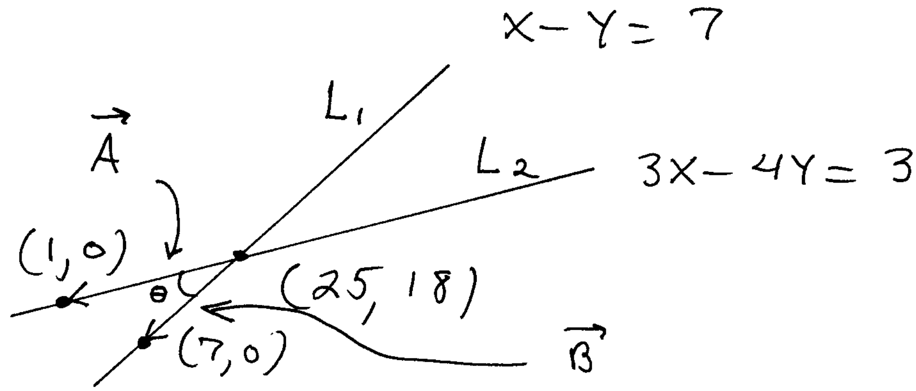


$$\cos 30° = \frac{x}{200} \rightarrow x = 200\cos 30°$$
$$= 200\left(\frac{\sqrt{3}}{2}\right) = 100\sqrt{3}\ \text{N};$$

Work = (force)(distance)

$$= (100\sqrt{3})(20) \approx 3464 \text{ joules}$$

51.)



$$\left.\begin{aligned} 3X - 4Y &= 3 \\ X - Y &= 7 \end{aligned}\right\} \quad \left.\begin{aligned} 3X - 4Y &= 3 \\ -4X + 4Y &= -28 \end{aligned}\right\}$$

$-X = -25 \rightarrow X = 25,\ Y = 18$ is pt. of intersection ; pt. $(7, 0)$ is on line L_1 and pt. $(1, 0)$ is on line L_2 ; form vectors $\vec{A}$ and $\vec{B}$ and find angle θ between them :

$\vec{A} = \overrightarrow{(-24, -18)}$ and $\vec{B} = \overrightarrow{(-18, -18)}$, so

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{432 + 324}{\sqrt{900}\cdot\sqrt{648}} = \frac{756}{30\sqrt{648}} \rightarrow$$

$$\theta = \arccos\left(\frac{756}{30\sqrt{648}}\right) \approx 8.13^\circ$$

Section 12.4

2.) $\vec{u} = \overrightarrow{(2,3,0)}$, $\vec{v} = \overrightarrow{(-1,1,0)}$ then

a.) $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \vec{k}$

$= (0)\vec{i} - (0)\vec{j} + 5\vec{k} = 5\vec{k}$;

$|\vec{u} \times \vec{v}| = 5$ and direction of $\vec{u} \times \vec{v}$ is $\vec{k}$.

b.) $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 2 & 3 & 0 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} \vec{k}$

$= (0)\vec{i} - (0)\vec{j} + (-5)\vec{k} = -5\vec{k}$;

$|\vec{v} \times \vec{u}| = 5$ and direction of $\vec{v} \times \vec{u}$ is $-\vec{k}$

3.) $\vec{u} = \overrightarrow{(2,-2,4)}$, $\vec{v} = \overrightarrow{(-1,1,-2)}$ then

a.) $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -2 \\ -1 & 1 \end{vmatrix} \vec{k}$

$= (0)\vec{i} - (0)\vec{j} + (0)\vec{k} = \overrightarrow{(0,0,0)}$;

$|\vec{u} \times \vec{v}| = 0$ and $\vec{u} \times \vec{v}$ has no direction

b.) $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} \vec{k}$$

$$= (0)\vec{i} - (0)\vec{j} + (0)\vec{k} = \overrightarrow{(0,0,0)} \; ;$$

$|\vec{v} \times \vec{u}| = 0$ and $\vec{v} \times \vec{u}$ has no direction.

7.) $\vec{u} = \overrightarrow{(-8,-2,-4)}$, $\vec{v} = \overrightarrow{(2,2,1)}$ then

a.) $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix}$

$$= \begin{vmatrix} -2 & -4 \\ 2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} -8 & -4 \\ 2 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} -8 & -2 \\ 2 & 2 \end{vmatrix} \vec{k}$$

$$= 6\vec{i} - (0)\vec{j} + (-12)\vec{k} = \overrightarrow{(6,0,-12)} \; ;$$

$|\vec{u} \times \vec{v}| = \sqrt{36+144} = \sqrt{180} = 6\sqrt{5}$ and

direction of $\vec{u} \times \vec{v}$ is $\frac{1}{6\sqrt{5}} \overrightarrow{(6,0,-12)} = \overrightarrow{\left(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right)}$.

b.) $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ -8 & -2 & -4 \end{vmatrix}$

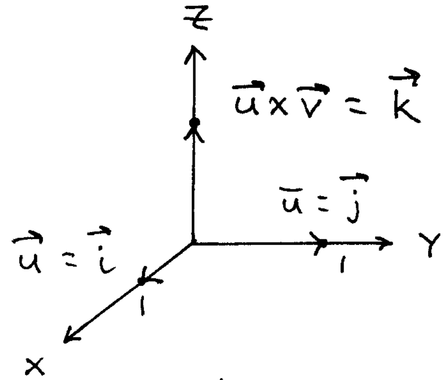
$$= \begin{vmatrix} 2 & 1 \\ -2 & -4 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 1 \\ -8 & -4 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 2 \\ -8 & -2 \end{vmatrix} \vec{k}$$

$$= (-6)\vec{i} - (0)\vec{j} + 12\vec{k} = \overrightarrow{(-6,0,12)} \; ;$$

$|\vec{v} \times \vec{u}| = \sqrt{180} = 6\sqrt{5}$ and direction

of $\vec{v} \times \vec{u}$ is $\frac{1}{6\sqrt{5}} \overrightarrow{(-6,0,12)} = \overrightarrow{\left(\frac{-1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)}$.

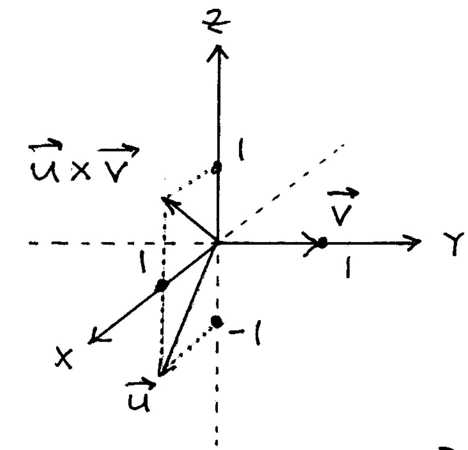
9.)



$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\vec{k}$$

$$= (0)\vec{i} - (0)j + (1)\vec{k} = \vec{k}$$

10.)

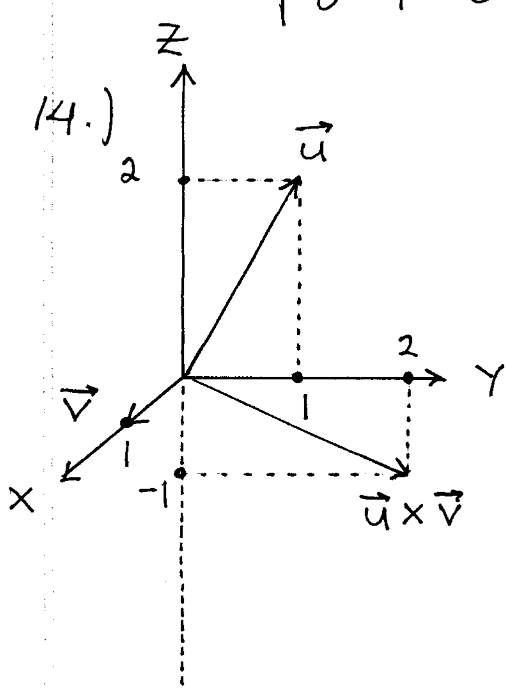


$\vec{u} = \vec{i} - \vec{k} = \overrightarrow{(1,0,-1)}$, $\quad \vec{V} = \vec{j} = \overrightarrow{(0,1,0)}$ then

$$\vec{u}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\vec{k}$$

$$= (1)\vec{i} - (0)\vec{j} + (1)\vec{k} = \vec{i} + \vec{k}$$

14.)



$\vec{u} = \vec{j} + 2\vec{k} = \overrightarrow{(0,1,2)}$,

$\vec{V} = \vec{i} = \overrightarrow{(1,0,0)}$, then

$$\vec{u}\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\vec{k}$$

$$= (0)\vec{i} - (-2)\vec{j} + (-1)\vec{k}$$

$$= 2\vec{j} - \vec{k}$$

16.) $P=(2,-2,1)$, $Q=(2,1,3)$, $R=(3,-1,1)$ so

$\vec{PQ} = \overrightarrow{(0,3,2)}$ and $\vec{PR} = \overrightarrow{(1,1,0)}$;

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 2 \\ 1 & 1 & 0 \end{vmatrix} = (-2)\vec{i} - (-2)\vec{j} + (-3)\vec{k}$$
$$= -2\vec{i} + 2\vec{j} - 3\vec{k} \; ;$$

area of Δ is $\frac{1}{2}$ area of $\square$ so

a.) $$\text{Area} = \frac{1}{2}\left|\vec{PQ}\times\vec{PR}\right|$$
$$= \frac{1}{2}\sqrt{4+4+9} = \frac{1}{2}\sqrt{17} \; ;$$

b.) a unit vector $\perp$ to plane PQR is

$$\vec{u} = \frac{1}{|\vec{PQ}\times\vec{PR}|} \cdot \vec{PQ}\times\vec{PR}$$
$$= \frac{1}{17}\overrightarrow{(-2,2,-3)} = \overrightarrow{\left(\frac{-2}{17}, \frac{2}{17}, \frac{-3}{17}\right)}$$

19.) $\vec{u} = 2\vec{i}$, $\vec{v} = 2\vec{j}$, $\vec{w} = 2\vec{k}$ then

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} = (0)\vec{i} - (0)\vec{j} + 4\vec{k} = 4\vec{k} \, ,$$

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 4\vec{i} - (0)\vec{j} + (0)\vec{k} = 4\vec{i} \, ,$$

$$\vec{w}\times\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{vmatrix} = (0)\vec{i} - (-4)\vec{j} + (0)\vec{k}$$
$$= 4\vec{j} \; ;$$

$(\vec{u} \times \vec{v}) \cdot \vec{w} = \overrightarrow{(0,0,4)} \cdot \overrightarrow{(0,0,2)} = 0+0+8 = 8,$

$(\vec{v} \times \vec{w}) \cdot \vec{u} = \overrightarrow{(4,0,0)} \cdot \overrightarrow{(2,0,0)} = 8+0+0 = 8,$

$(\vec{w} \times \vec{u}) \cdot \vec{v} = \overrightarrow{(0,4,0)} \cdot \overrightarrow{(0,2,0)} = 0+8+0 = 8;$

volume of parallelepiped is

$\text{Volume} = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = |8| = 8$

20.) $\vec{u} = \overrightarrow{(1,-1,1)}$, $\vec{v} = \overrightarrow{(2,1,-2)}$, $\vec{w} = \overrightarrow{(-1,2,-1)}$ then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{vmatrix} = (1)\vec{i} - (-4)\vec{j} + (3)\vec{k} = \vec{i} + 4\vec{j} + 3\vec{k},$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = (3)\vec{i} - (-4)\vec{j} + (5)\vec{k} = 3\vec{i} + 4\vec{j} + 5\vec{k},$$

$$\vec{w} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = (1)\vec{i} - (0)\vec{j} + (-1)\vec{k} = \vec{i} - \vec{k};$$

$(\vec{u} \times \vec{v}) \cdot \vec{w} = \overrightarrow{(1,4,3)} \cdot \overrightarrow{(-1,2,-1)} = -1+8-3 = 4,$

$(\vec{v} \times \vec{w}) \cdot \vec{u} = \overrightarrow{(3,4,5)} \cdot \overrightarrow{(1,-1,1)} = 3-4+5 = 4,$

$(\vec{w} \times \vec{u}) \cdot \vec{v} = \overrightarrow{(1,0,-1)} \cdot \overrightarrow{(2,1,-2)} = 2+0+2 = 4;$

volume of parallelepiped is

$\text{Volume} = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = |4| = 4$

23.) $\vec{u} = 5\vec{i} - \vec{j} + \vec{k}$, $\vec{v} = \vec{j} - 5\vec{k}$,

$\vec{w} = -15\vec{i} + 3\vec{j} - 3\vec{k} = -3(5\vec{i} - \vec{j} + \vec{k}) = -3\vec{u}$:

a.) $\vec{u}\cdot\vec{v} = (5)(0)+(-1)(1)+(1)(-5) = -6$,

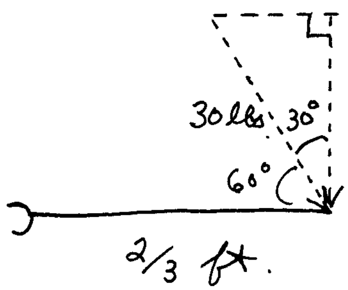
$\vec{v}\cdot\vec{w} = \vec{v}\cdot(-3\vec{u}) = -3(\vec{u}\cdot\vec{v}) = -3(-6) = 18$,

$\vec{u}\cdot\vec{w} = \vec{u}\cdot(-3\vec{u}) = -3\left((5)^2+(-1)^2+(1)^2\right) = -81$;

so no vectors are $\perp$

b.) $\vec{w} = -3\vec{u}$ so $\vec{u}$ and $\vec{w}$ are $\parallel$.

25.)



$\frac{\sqrt{3}}{2}\cdot(30) = 15\sqrt{3}$ lbs.

Torque = (vert. force)(distance)

$= (15\sqrt{3})(2/3) = 10\sqrt{3}$ ft.-lbs.

27.)
a.) True
b.) False
c.) True
d.) True
e.) False
f.) True
g.) True
h.) True

28.)
a.) True
b.) True
c.) True
d.) True
e.) True
f.) True
g.) True
h.) True

29.) a.) $\text{proj}_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$

b.) $\vec{u}\times\vec{v}$ is $\perp$ to $\vec{u}$ and $\vec{v}$

c.) $(\vec{u}\times\vec{v})\times\vec{w}$ is $\perp$ to $\vec{u}\times\vec{v}$ and $\vec{w}$

d.) Volume of parallelepiped is $|(\vec{u}\times\vec{v})\cdot\vec{w}|$

31.) a.) Sense
b.) Not
c.) Sense
d.) Not

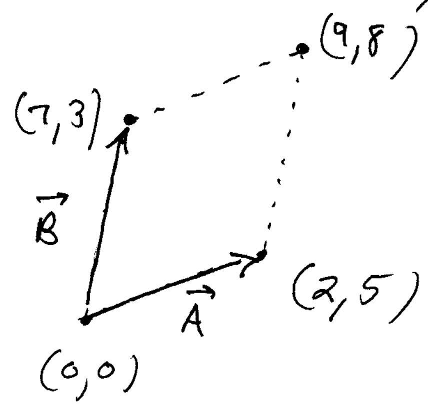
33.) Not necessarily: Let $\vec{v} = \vec{0}$ and $\vec{w} = 2\vec{u}$. Then

$$\vec{u} \times \vec{v} = \vec{u} \times \vec{0} = \vec{0} \quad \text{and}$$

$$\vec{u} \times \vec{w} = \vec{u} \times (2\vec{u}) = 2(\vec{u} \times \vec{u}) = 2(\vec{0}) = \vec{0},$$

but $\vec{v} \neq \vec{w}$.

36.) $\vec{A} = \overrightarrow{(2,5,0)}$, $\vec{B} = \overrightarrow{(7,3,0)}$, so area of parallelogram is $|\vec{A} \times \vec{B}|$:

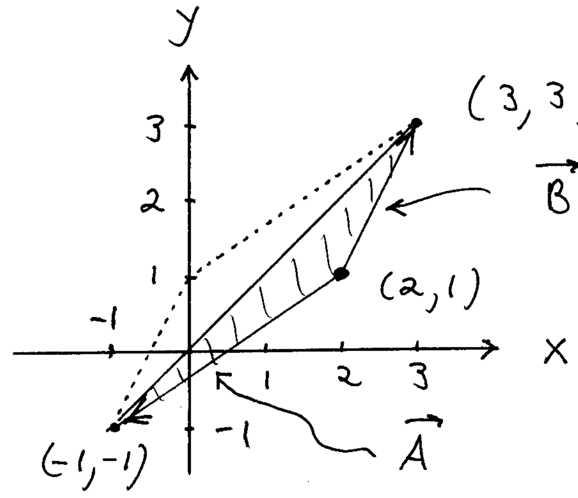


$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 5 & 0 \\ 7 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 5 & 0 \\ 3 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 5 \\ 7 & 3 \end{vmatrix} \vec{k}$$

$$= (0-0)\vec{i} - (0-0)\vec{j} + (-29)\vec{k} = -29\vec{k},$$

so Area $= |\vec{A} \times \vec{B}| = |-29\vec{k}| = 29$

40.)



area of triangle is $\frac{1}{2}$ area of parallelogram ; so

$\vec{A} = \overrightarrow{(-3, -2, 0)}$, $\vec{B} = \overrightarrow{(1, 2, 0)}$,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 0 \\ 2 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & 0 \\ 1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & -2 \\ 1 & 2 \end{vmatrix} \vec{k}$$

$$= (0-0)\vec{i} - (0-0)\vec{j} + (-4)\vec{k} = -4\vec{k} \;;$$

$$\text{Area of } \Delta = \frac{1}{2} |\vec{A} \times \vec{B}|$$

$$= \frac{1}{2} |-4\vec{k}|$$

$$= \frac{1}{2}(4)$$

$$= 2 \; .$$

Section 12.5

1.) point $P = (3, -4, -1)$, vector $\vec{A} = \overrightarrow{(1,1,1)}$, so

line L: $\begin{cases} X = 3 + (1)t = 3 + t \\ Y = -4 + (1)t = -4 + t \\ Z = -1 + (1)t = -1 + t \end{cases}$

4.) points $P = (1, 2, 0)$, $Q = (1, 1, -1)$ and vector $\overrightarrow{PQ} = \overrightarrow{(0, -1, -1)}$, so line

$L: \begin{cases} x = 1 + (0)t = 1 \\ y = 2 + (-1)t = 2 - t \\ z = 0 + (-1)t = -t \end{cases}$

6.) point $P = (3, -2, 1)$ and $\parallel$ to line

$L: \begin{cases} x = 1 + 2t \\ y = 2 - t \\ z = 3t \end{cases}$ so $\parallel$ vector is $\vec{A} = \overrightarrow{(2, -1, 3)}$

and line is

$M: \begin{cases} x = 3 + (2)t = 3 + 2t \\ y = -2 + (-1)t = -2 - t \\ z = 1 + (3)t = 1 + 3t \end{cases}$

7.) point $P = (1, 1, 1)$ and $\parallel$ to z-axis so $\parallel$ vector is $\vec{A} = \overrightarrow{(0, 0, 1)}$, and line is

$L: \begin{cases} x = 1 + (0)t = 1 \\ y = 1 + (0)t = 1 \\ z = 1 + (1)t = 1 + t \end{cases}$

8.) point $P = (2, 4, 5)$ and $\perp$ to plane

$3X + 7Y - 5Z = 21$; plane has $\perp$ vector $\vec{A} = \overrightarrow{(3,7,-5)}$, so line is

$$L: \begin{cases} X = 2 + (3)t = 2 + 3t \\ Y = 4 + (7)t = 4 + 7t \\ Z = 5 + (-5)t = 5 - 5t \end{cases}$$

10.) $\vec{u} = \overrightarrow{(1,2,3)}$, $\vec{V} = \overrightarrow{(3,4,5)}$ so

$$\vec{u} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = (10-12)\vec{i} - (5-9)\vec{j} + (4-6)\vec{k}$$

$= -2\vec{i} + 4\vec{j} - 2\vec{k}$ is $\perp$ to $\vec{u}$ and $\vec{V}$, and point $P = (2,3,0)$ so line

$$L: \begin{cases} X = 2 + (-2)t = 2 - 2t \\ Y = 3 + (4)t = 3 + 4t \\ Z = 0 + (-2)t = -2t \end{cases}$$

21.) point $P = (0,2,-1)$ and $\perp$ vector $\vec{u} = \overrightarrow{(3,-2,-1)}$, so plane is

$3(X-0) - 2(Y-2) - 1(Z-(-1)) = 0 \rightarrow$

$3X - 2Y + 4 - Z - 1 = 0 \rightarrow$

$3X - 2Y - Z = -3$

22.) plane $3X + Y + Z = 7$ has $\perp$ vector $\vec{u} = (3,1,1)$, and point $P = (1,-1,3)$, so new plane is

$3(X-1) + 1\cdot(Y-(-1)) + 1\cdot(Z-3) = 0 \rightarrow$

$3X - 3 + Y + 1 + Z - 3 = 0 \rightarrow$

$3X + Y + Z = 5$

24.) points $P = (2, 4, 5)$, $Q = (1, 5, 7)$, $R = (-1, 6, 8)$ so vectors $\overrightarrow{PQ} = \overrightarrow{(-1, 1, 2)}$ and $\overrightarrow{PR} = \overrightarrow{(-3, 2, 3)}$, so $\perp$ vector to plane is

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = (3-4)\vec{i} - (-3+6)\vec{j} + (-2+3)\vec{k}$$

$= -\vec{i} - 3\vec{j} + \vec{k}$; equation of plane is $-1(x-2) - 3(Y-4) + 1(Z-5) = 0 \rightarrow$

$-x + 2 - 3Y + 12 + Z - 5 = 0 \rightarrow$

$-x - 3Y + Z = -9$

25.) Line $L: \begin{cases} x = 5 + t \\ Y = 1 + 3t \\ Z = 4t \end{cases}$ has $\parallel$ vector $\vec{A} = \overrightarrow{(1, 3, 4)}$, so plane has $\perp$ vector $\vec{A} = \overrightarrow{(1, 3, 4)}$, so plane through point $P = (2, 4, 5)$ is

$1 \cdot (x-2) + 3 \cdot (Y-4) + 4 \cdot (Z-5) = 0 \rightarrow$

$x - 2 + 3Y - 12 + 4Z - 20 = 0 \rightarrow$

$x + 3Y + 4Z = 34$

28.) Lines $L_1: \begin{cases} x = t \\ y = 2 - t \\ z = 1 + t \end{cases}$ and $L_2: \begin{cases} x = 2 + 2s \\ y = 3 + s \\ z = 6 + 5s \end{cases}$;

if lines intersect, then

$$\left.\begin{aligned} t &= 2 + 2s \\ 2 - t &= 3 + s \end{aligned}\right\} \text{(add)} \rightarrow 2 = 5 + 3s \rightarrow$$

$s = -1$ and $t = 0$, so pt. of $\cap$ is $\boxed{P = (x, y, z) = (0, 2, 1)}$; vector $\parallel$ to L_1 is $\vec{A} = \overrightarrow{(1, -1, 1)}$, vector $\parallel$ to L_2 is $\vec{B} = \overrightarrow{(2, 1, 5)}$, so vector $\perp$ to plane is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = (-5-1)\vec{i} - (5-2)\vec{j} + (1+2)\vec{k}$$

$= \underline{\underline{-6\vec{i} - 3\vec{j} + 3\vec{k}}}$; then equation of plane is

$-6 \cdot (x - 0) - 3(y - 2) + 3(z - 1) = 0 \rightarrow$

$-6x - 3y + 6 + 3z - 3 = 0 \rightarrow$

$-6x - 3y + 3z = -3 \rightarrow$

$2x + y - z = 1$.

31.) plane $2x + y - z = 3$ has $\perp$ vector $\vec{A} = \overrightarrow{(2, 1, -1)}$; plane $x + 2y + z = 2$ has $\perp$ vector $\vec{B} = \overrightarrow{(1, 2, 1)}$, so plane $\parallel$ to the line of intersection of these planes is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (1+2)\,\vec{i} - (2+1)\,\vec{j} + (4-1)\,\vec{k}$$

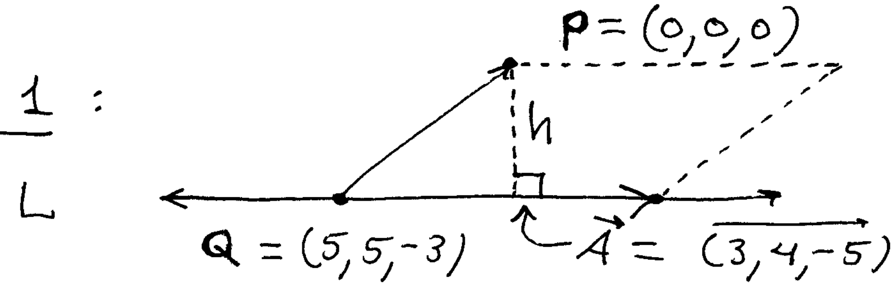
$$= \underline{\underline{3\vec{i} - 3\vec{j} + 3\vec{k}}}$$; so plane $\perp$ to $\vec{A} \times \vec{B}$ and through point $P = (2,1,-1)$ is

$$3(x-2) - 3(y-1) + 3(z+1) = 0 \rightarrow$$
$$(x-2) - (y-1) + (z+1) = 0 \rightarrow$$
$$x - y + z = 0$$

34.) point $P = (0,0,0)$, line

$$L: \begin{cases} x = 5 + 3t \\ y = 5 + 4t \\ z = -3 - 5t \end{cases}$$; distance from point P to line L is :

<u>Method 1</u> :



Point $Q = (5,5,-3)$ is on line L and vector $\vec{A} = \overrightarrow{(3,4,-5)}$ is $\parallel$ to L ; vector $\overrightarrow{QP} = \overrightarrow{(-5,-5,3)}$; area of parallelogram formed by $\overrightarrow{QP}$ and $\vec{A}$ is

$$|\overrightarrow{QP} \times \vec{A}| = (\text{base})(\text{height})$$
$$= |\vec{A}| \cdot h \Rightarrow$$

$$h = \frac{|\vec{QP} \times \vec{A}|}{|\vec{A}|} \quad ; \quad \vec{QP} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix}$$

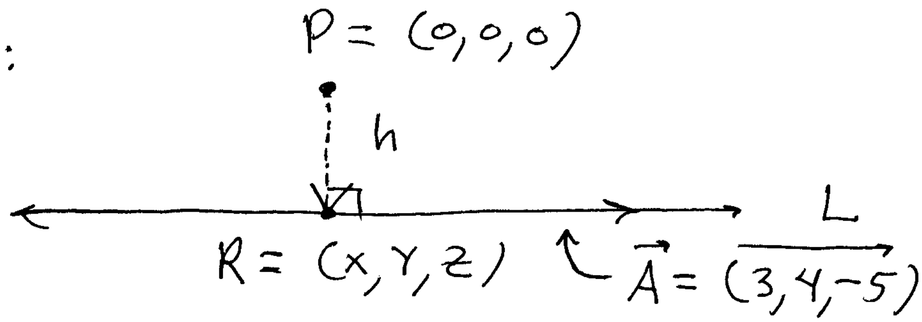
$$= (25-12)\vec{i} - (25-9)\vec{j} + (-20+15)\vec{k}$$

$$= 13\vec{i} - 16\vec{j} - 5\vec{k} \quad , \text{ so}$$

$$|\vec{QP} \times \vec{A}| = \sqrt{169+256+25} = \sqrt{450} = 15\sqrt{2} \; ;$$

then $$h = \frac{|\vec{QP} \times \vec{A}|}{|\vec{A}|} = \frac{15\sqrt{2}}{\sqrt{50}} = \frac{15\sqrt{2}}{5\sqrt{2}} = 3$$

<u>Method 2</u>:



Find point $R = (x,y,z)$ on line L so that vector $\vec{PR} = \overrightarrow{(x,y,z)}$ is $\perp$ to $\vec{A}$ $\Rightarrow$ $\vec{PR} \cdot \vec{A} = 0$ $\Rightarrow$

$$\overrightarrow{(x,y,z)} \cdot \overrightarrow{(3,4,-5)} = 0 \Rightarrow$$

$$3x + 4y - 5z = 0 \Rightarrow$$

$$3(5+3t) + 4(5+4t) - 5(-3-5t) = 0 \Rightarrow$$

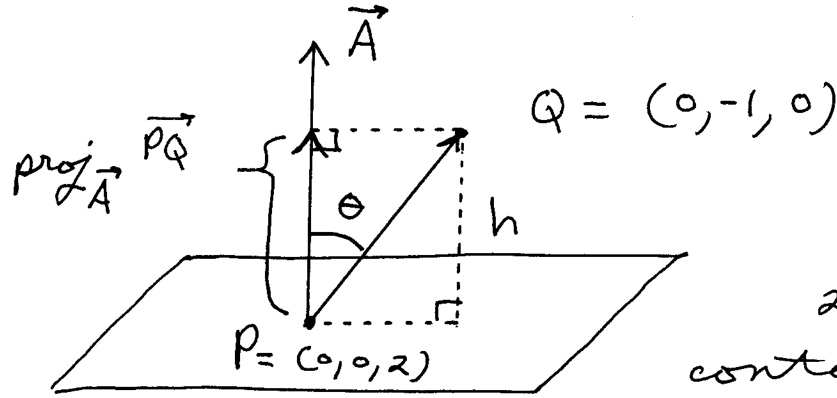
$$15 + 9t + 20 + 16t + 15 + 25t = 0 \Rightarrow$$

$$50t + 50 = 0 \longrightarrow \underline{\underline{t = -1}} \; ; \text{ then}$$

point $R = (2,1,2)$ and distance from point P to R is

$$h = \sqrt{(2-0)^2 + (1-0)^2 + (2-0)^2} = \sqrt{9} = 3$$

43.)



plane $2X + Y + 2Z = 4$ contains the point $P = (0,0,2)$; vector $\vec{A} = \overrightarrow{(2,1,2)}$ is $\perp$ to plane; then distance from Q to plane is

$$h = \left| proj_{\vec{A}} \vec{PQ} \right|$$

$$= |\vec{PQ}| \cdot |\cos\theta|$$

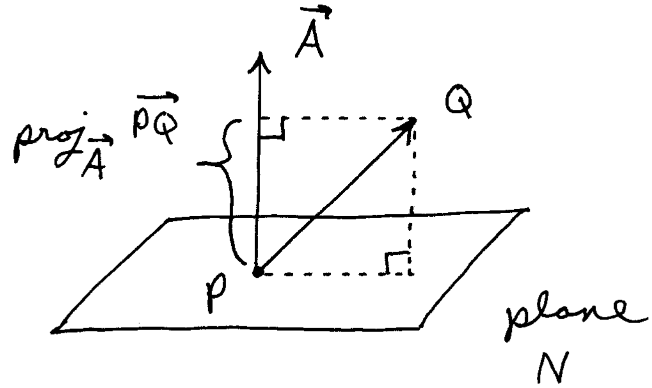
$$= \cancel{|\vec{PQ}|} \cdot \frac{|\vec{PQ} \cdot \vec{A}|}{\cancel{|\vec{PQ}|}\,|\vec{A}|}$$

$$= \frac{\left| \overrightarrow{(0,-1,-2)} \cdot \overrightarrow{(2,1,2)} \right|}{\left| \overrightarrow{(2,1,2)} \right|}$$

$$= \frac{|0-1-4|}{\sqrt{9}} = 5/3$$

45.) Planes $M: X + 2Y + 6Z = 1$ and $N: X + 2Y + 6Z = 10$ are $\parallel$; point $Q = (1,0,0)$ is on M and point $P = (10,0,0)$ is on N; vector $\vec{A} = \overrightarrow{(1,2,6)}$ is $\perp$ to N; now find distance from point Q to plane N; then distance

$$h = \left| proj_{\vec{A}} \vec{PQ} \right|$$



$$= |\vec{PQ}|\,|\cos\theta|$$

$$= \cancel{|\vec{PQ}|}\cdot\frac{|\vec{PQ}\cdot\vec{A}|}{\cancel{|\vec{PQ}|}\,|\vec{A}|}$$

$$= \frac{|\overrightarrow{(-9,0,0)}\cdot\overrightarrow{(1,2,6)}|}{|\overrightarrow{(1,2,6)}|} = \frac{|-9|}{\sqrt{41}} = \frac{9}{\sqrt{41}}$$

47.) Plane $X+Y=1$ has $\perp$ vector $\vec{A}=\overrightarrow{(1,1,0)}$; plane $2X+Y-2Z=2$ has $\perp$ vector $\vec{B}=\overrightarrow{(2,1,-2)}$; the angle between the planes is the angle (acute : $0° \le \theta \le 90°$) determined by the $\perp$ vectors :

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{2+1-0}{\sqrt{2}\cdot\sqrt{9}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow$$

$$\theta = 45°$$

55.) plane $X+Y+Z=2$, line $L: \begin{cases} x = 1+2t \\ y = 1+5t \\ z = 3t \end{cases}$;

the point of $\cap$ is given by

$X+Y+Z = (1+2t)+(1+5t)+(3t) = 2 \Rightarrow$

$10t = 0 \rightarrow t = 0 \rightarrow$ point is

$(x,y,z) = (1,1,0)$

59.) Plane $x-2y+4z=2$ has $\perp$ vector $\vec{A} = \overrightarrow{(1,-2,4)}$; plane $x+y-2z=5$ has $\perp$ vector $\vec{B} = \overrightarrow{(1,1,-2)}$; then the line forming the $\cap$ of these planes is $\parallel$ to the vector

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = (4-4)\vec{i} - (-2-4)\vec{j} + (1+2)\vec{k}$$

$$= \underline{\underline{6\vec{j} + 3\vec{k}}}$$; now find a point on this line :

$$\left.\begin{array}{l} x-2y+4z=2 \\ x+y-2z=5 \end{array}\right\} \quad \left.\begin{array}{l} \underline{x = 2+2y-4z} \\ x = 5-y+2z \end{array}\right\}$$

$2+2y-4z = 5-y+2z \rightarrow$

$3y = 3+6z \longrightarrow \underline{y=1+2z}$; now let z be ANY number : $z=0 \rightarrow$

$y=1 \longrightarrow x=4$, so point

$(x,y,z) = \underline{\underline{(4,1,0)}}$ lies on BOTH planes;

now the line of intersection is given by

$$L: \begin{cases} x = 4+(0)t = 4 \\ y = 1+(6)t = 1+6t \\ z = 0+(3)t = 3t \end{cases}$$

67.) Line $L: \begin{cases} x = 1-2t \\ y = 2+5t \\ z = -3t \end{cases}$ has $\parallel$ vector

$\vec{A} = \overrightarrow{(-2, 5, -3)}$; plane $2X + Y - Z = 8$ has $\perp$ vector $\vec{B} = \overrightarrow{(2, 1, -1)}$; then line and plane are parallel if and only if $\vec{A} \perp \vec{B}$:

$\vec{A} \cdot \vec{B} = -4 + 5 + 3 = 4 \neq 0$ so $\vec{A}$ not $\perp$ $\vec{B}$ and line is NOT parallel to plane .