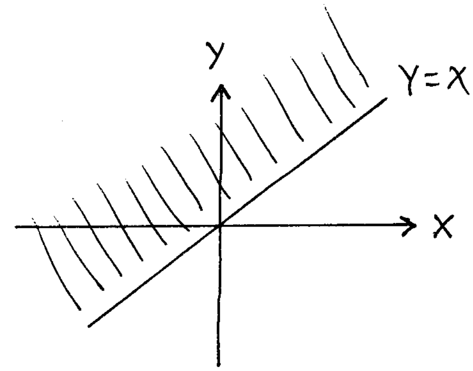
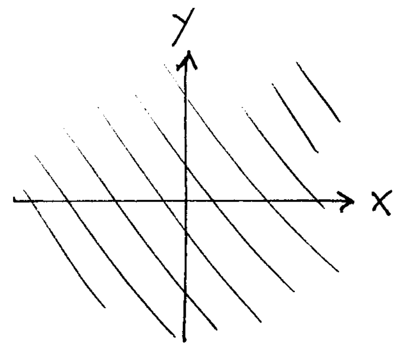
Section 14.1



2.) $f(x,y) = \sqrt{y-x}$

a.) Domain : $y - x \geq 0$.
so all pts. (x,y) with
$y \geq x$

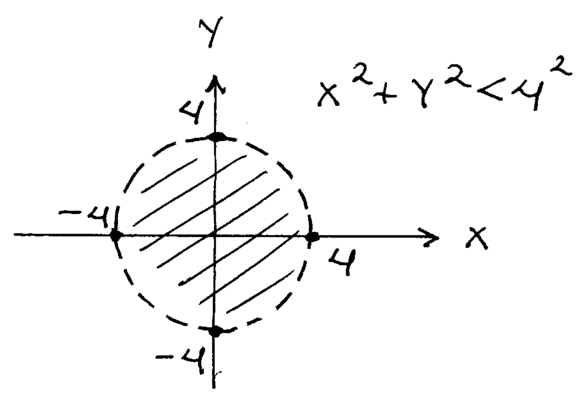
b.) Consider all pts. $(0,y)$ for $0 \leq y < \infty$;
for these pts. the z-value is
$z = \sqrt{y}$ and $0 \leq z < \infty$. It follows
(since $\sqrt{x-y} \geq 0$) that the Range of f
is $0 \leq z < \infty$.



3.) $f(x,y) = 4x^2 + 9y^2$

a.) Domain :
all pts. (x,y)

b.) Consider all pts.
$(x,0)$ for $-\infty < x < \infty$; for these pts. the
z value is $z = 4x^2$ and $0 \leq z < \infty$.
It follows (since $4x^2 + 9y^2 \geq 0$) that
the Range of f is $0 \leq z < \infty$.



7.) $f(x,y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$

a.) Domain : $16 - x^2 - y^2 > 0$
$\rightarrow$ $x^2 + y^2 < 16$ so
domain is set of pts.
(x,y) <u>inside</u> the circle

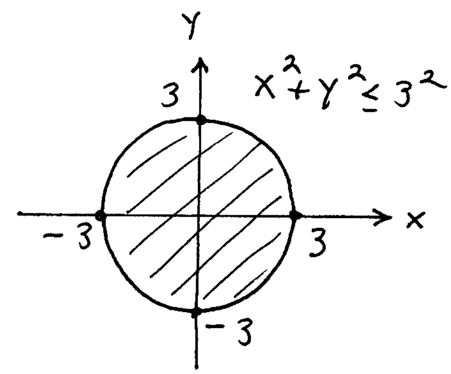
$$x^2 + y^2 = 4^2$$

b.) Consider all pts. $(x,0)$, where $-4<x<4$; for these pts. the z-value is $z = \dfrac{1}{\sqrt{16-x^2}}$; note that $z = \frac{1}{4}$ if $x=0$ and

$$\lim_{x \to 4^-} z = \lim_{x \to 4^-} \frac{1}{\sqrt{16-x^2}} = \frac{1}{\sqrt{0}} = +\infty \ ;$$

the z-values range for $z = \frac{1}{4}$ to ∞; since $\frac{1}{4} \le \dfrac{1}{\sqrt{16-x^2-y^2}}$, it follows that the Range of f is $\frac{1}{4} \le z < \infty$.

8.) $f(x,y) = \sqrt{9-x^2-y^2}$

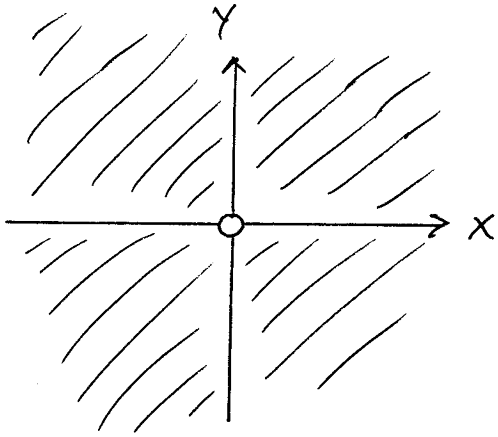


a.) Domain: $9-x^2-y^2 \ge 0$ $\rightarrow$ $x^2+y^2 \le 9$ so domain is set of all pts. (x,y) on or inside the circle $x^2+y^2 = 3^2$

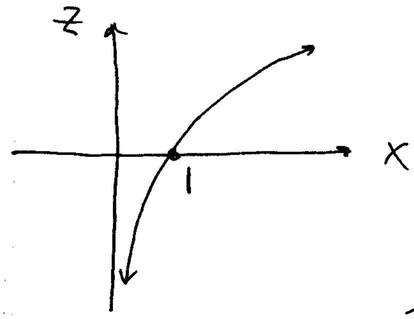
b.) Consider that $z = \sqrt{9-x^2-y^2}$ $\rightarrow$ $z^2 = 9-x^2-y^2$ $\rightarrow$ $x^2+y^2+z^2 = 3^2$ is a sphere of radius 3 centered at $(0,0,0)$; the $z = \sqrt{9-x^2-y^2}$ is the top half of the sphere, so the Range of f is $0 \le z \le 3$

9.) $f(x,y) = \ln(x^2+y^2)$

a.) Domain: $x^2+y^2 > 0$ so domain is set of all pts. (x,y) except $(0,0)$;



b.) Consider all pts. $(x, 0)$, where $0 < x < \infty$; for these pts. the z-value is $z = \ln x^2 \rightarrow z = 2 \ln x$;



these z-values range from $-\infty$ to $+\infty$; it follows that the Range of f is $-\infty < z < \infty$.

Section 14.2

1.) $\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2} = \frac{0-0+5}{0+0+2} = \frac{5}{2}$

4.) $\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 = \left(\frac{1}{2}+\frac{-1}{3}\right)^2 = \frac{1}{36}$

9.) $\lim_{(x,y)\to(0,0)} \frac{e^y \cdot \sin x}{x} = \lim_{(x,y)\to(0,0)} e^y \cdot \left(\frac{\sin x}{x}\right)$

$= e^0 \cdot (1) = (1)(1) = 1$

12.) $\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y+1}{y-\sin x} = \frac{\cos 0 + 1}{0 - \sin\frac{\pi}{2}} = \frac{1+1}{-1} = -2$

14.) $\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{(x,y)\to(1,1)} \frac{\cancel{(x-y)}(x+y)}{\cancel{(x-y)}} = 1+1 = 2$

16.) $\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y-xy+4x^2-4x}$

$= \lim_{(x,y)\to(2,-4)} \frac{y+4}{xy(x-1)+4x(x-1)}$

$= \lim_{(x,y)\to(2,-4)} \frac{y+4}{(x-1)[xy+4x]}$

$= \lim_{(x,y)\to(2,-4)} \frac{\cancel{y+4}}{x(x-1)[\cancel{y+4}]} = \frac{1}{2(1)} = \frac{1}{2}$

20.) $\lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} \cdot \frac{\sqrt{x}+\sqrt{y+1}}{\sqrt{x}+\sqrt{y+1}}$

$$= \lim_{(x,y)\to(4,3)} \frac{x-(y+1)}{(x-y-1)(\sqrt{x}+\sqrt{y+1})} = \lim_{(x,y)\to(4,3)} \frac{\cancel{x-y-1}}{\cancel{(x-y-1)}(\sqrt{x}+\sqrt{y+1})}$$

$$= \frac{1}{2+2} = \frac{1}{4}$$

35.) $\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$ DNE since

<u>along path $y=0$</u>:

$$\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2}}$$

$$= \lim_{x\to 0} \frac{-x}{|x|} = \begin{cases} \lim_{x\to 0} \frac{-x}{x} = \lim_{x\to 0} -1 = -1, & \text{if } x>0 \\ \lim_{x\to 0} \frac{-x}{-x} = \lim_{x\to 0} 1 = 1, & \text{if } x<0 \end{cases}$$

so $\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$ DNE along path $y=0$.

36.) $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ DNE since

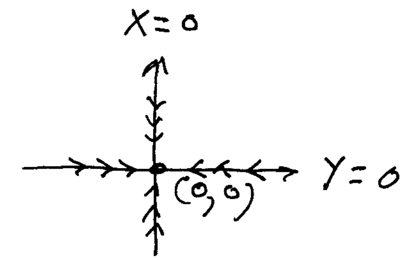
<u>along path $y=0$</u>:

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{x^4} = \lim_{x\to 0} 1 = 1;$$

<u>along path $x=0$</u>:

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{0+y^2} = \lim_{y\to 0} 0 = 0.$$

37.) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$ DNE since



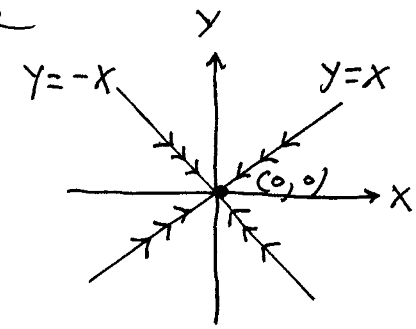
Along path Y=0:

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{x^4} = \lim_{x\to 0} 1 = 1 \;;$$

Along path X=0:

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = \lim_{y\to 0} -1 = -1$$

38.) $\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|}$ DNE since



Along path Y=X:

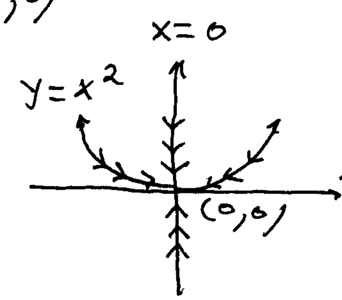
$$\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|} = \lim_{(x,y)\to(0,0)} \frac{x^2}{|x^2|}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = \lim_{(x,y)\to(0,0)} 1 = 1 \;;$$

Along path Y=-X:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|} = \lim_{(x,y)\to(0,0)} \frac{-x^2}{x^2} = \lim_{(x,y)\to(0,0)} -1 = -1 .$$

41.) $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y}$ DNE since



Along path X=0:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y} = \lim_{(x,y)\to(0,0)} \frac{y}{y} = \lim_{y\to 0} 1 = 1 \;;$$

Along path $Y=X^2$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y} = \lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2} = \lim_{x\to 0} 2 = 2 .$$

Chapter 14
Practice Exercises

12.) $\lim_{(x,y)\to(1,1)} \frac{x^3y^3-1}{xy-1} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{(x,y)\to(1,1)} \frac{(xy)^3-(1)^3}{xy-1}$

$= \lim_{(x,y)\to(1,1)} \frac{\cancel{(xy-1)}((xy)^2+(xy)+1)}{\cancel{xy-1}}$

$= 1+1+1 = 3$

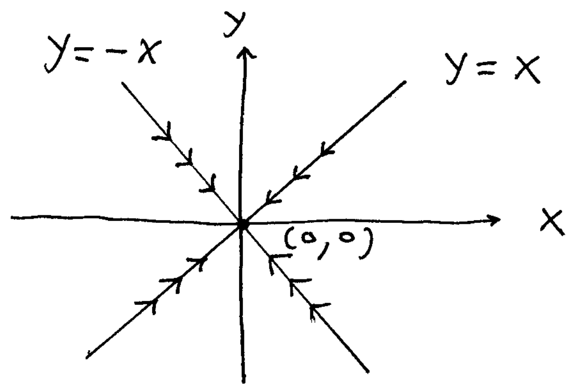
16.) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$ DNE since

along path $y=x$:

$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} = \lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2} = \lim_{x\to 0} 2 = 2$;

along path $y=-x$:

$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} = \lim_{(x,y)\to(0,0)} \frac{2x^2}{-x^2} = \lim_{x\to 0} -2 = -2$.



Section 4.3

2.) $f(x,y) = x^2 - xy + y^2 \longrightarrow$

$$\frac{\partial f}{\partial x} = 2x - y \ , \quad \frac{\partial f}{\partial y} = -x + 2y$$

7.) $f(x,y) = (x^2+y^2)^{1/2} \longrightarrow$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}} \ ,$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

10.) $f(x,y) = \dfrac{x}{x^2+y^2} \longrightarrow$

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \ ,$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

13.) $f(x,y) = e^{x+y+1} \longrightarrow$

$$\frac{\partial f}{\partial x} = e^{x+y+1}\cdot(1) \ , \quad \frac{\partial f}{\partial y} = e^{x+y+1}\cdot(1)$$

16.) $f(x,y) = e^{xy}\ln y \longrightarrow$

$$\frac{\partial f}{\partial x} = e^{xy}\cdot(y)\cdot\ln y + e^{xy}\cdot(0) = ye^{xy}\ln y \ ,$$

$$\frac{\partial f}{\partial y} = e^{xy}\cdot\frac{1}{y} + xe^{xy}\cdot\ln y = e^{xy}\left(\frac{1}{y} + x\ln y\right)$$

21.) $f(x,y) = \int_x^y g(t)\,dt \longrightarrow f(x,y) = -\int_y^x g(t)\,dt$;

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(-\int_y^x g(t)\,dt\right) = -g(x) \quad (\text{by FTC 1}) \ ,$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int_x^y g(t)\,dt\right) = g(y) \quad (\text{by FTC I})$$

43.) $g(x,y) = x^2 y + \cos y + y \sin x \longrightarrow$

$g_x = 2xy + y\cos x$, $g_y = x^2 - \sin y + \sin x$,

$g_{xx} = 2y - y\sin x$, $g_{yy} = -\cos y$,

$g_{xy} = 2x + \cos x$, $g_{yx} = 2x + \cos x$

45.) $r(x,y) = \ln(x+y) \longrightarrow$

$r_x = \dfrac{1}{x+y}$, $r_y = \dfrac{1}{x+y}$,

$r_{xx} = \dfrac{-1}{(x+y)^2}$, $r_{yy} = \dfrac{-1}{(x+y)^2}$,

$r_{xy} = \dfrac{-1}{(x+y)^2}$, $r_{yx} = \dfrac{-1}{(x+y)^2}$

46.) $s(x,y) = \arctan\left(\dfrac{y}{x}\right) \longrightarrow$

$$s_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2},$$

$$s_y = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{x+\frac{y^2}{x}} \cdot \frac{x}{x} = \frac{x}{x^2+y^2},$$

$$s_{xx} = \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2},$$

$$s_{yy} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2},$$

$$S_{xy} = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$S_{yx} = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

54.) $f(x,y) = 4+2x-3y-xy^2$;

$$\frac{\partial f}{\partial x}(-2,1) = \lim_{h\to 0} \frac{f(-2+h,1)-f(-2,1)}{h}$$

$$= \lim_{h\to 0} \frac{[4+2(-2+h)-3(1)-(-2+h)(1)^2]-(-1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}+2h-\cancel{3}+\cancel{2}-h+\cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1 \quad ;$$

$$\frac{\partial f}{\partial y}(-2,1) = \lim_{h\to 0} \frac{f(-2,1+h)-f(-2,1)}{h}$$

$$= \lim_{h\to 0} \frac{[4+2(-2)-3(1+h)-(-2)(1+h)^2]-(-1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}-3-3h+2(1+2h+h^2)+1}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{3}-3h+\cancel{2}+4h+2h^2+\cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{2h^2+h}{h}$$

$$= \lim_{h\to 0} (2h+1) = 1 .$$

57.) assume $z = f(x,y)$ and $xy+z^3x-2yz=0$

$\rightarrow \frac{\partial}{\partial x}\left(xy + z^3x - 2yz\right) = \frac{\partial}{\partial x}(0)$

$\rightarrow y + \left(z^3 \cdot (1) + \left(3z^2 \cdot z_x\right) \cdot x\right) - 2y z_x = 0$

$\rightarrow y + z^3 + 3xz^2 \cdot z_x - 2y z_x = 0$

$\rightarrow (3xz^2 - 2y) z_x = -y - z^3$

$\rightarrow z_x = \dfrac{-y - z^3}{3xz^2 - 2y}$ (Let $x=1$, $y=1$, $z=1$)

$\rightarrow z_x = \dfrac{-1-1}{3-2} = -2$

65.) Show $f(x,y) = e^{-2y}\cos 2x$ satisfies $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$:

$f_x = e^{-2y} \cdot -\sin 2x \cdot 2 = -2\sin 2x \cdot e^{-2y}$,

$f_y = -2e^{-2y} \cdot \cos 2x = -2\cos 2x \cdot e^{-2y}$,

$f_{xx} = -2 \cdot (2\cos 2x) \cdot e^{-2y} = -4\cos 2x \cdot e^{-2y}$,

$f_{yy} = -2\cos 2x \cdot \left(-2e^{-2y}\right) = 4\cos 2x \cdot e^{-2y}$;

then $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \left(-4\cos 2x \cdot e^{-2y}\right) + \left(4\cos 2x \cdot e^{-2y}\right) = 0$

66.) Show $\ln (x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2)$ satisfies $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$:

$f_x = \frac{1}{2} \cdot \dfrac{1}{x^2+y^2} \cdot 2x = \dfrac{x}{x^2+y^2}$,

$f_y = \frac{1}{2} \cdot \dfrac{1}{x^2+y^2} \cdot 2y = \dfrac{y}{x^2+y^2}$,

$$f_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} ,$$

$$f_{yy} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2} \text{ ; then}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$= \frac{\cancel{y^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0 .$$

69.) Show $w = \sin(x+ct)$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = c^2 \cdot \frac{\partial^2 w}{\partial x^2} :$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) \cdot c = c \cdot \cos(x+ct) ,$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) \cdot (1) = \cos(x+ct) ,$$

$$\frac{\partial^2 w}{\partial t^2} = c \cdot -\sin(x+ct) \cdot c = -c^2 \sin(x+ct) ,$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) \cdot (1) = -\sin(x+ct) ;$$

then $$\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct)$$

$$= c^2 (-\sin(x+ct))$$

$$= c^2 \cdot \frac{\partial^2 w}{\partial x^2} .$$

72.) Show $w = \ln(2x + 2ct)$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = c^2 \cdot \frac{\partial^2 w}{\partial x^2} \quad :$$

$$\frac{\partial w}{\partial t} = \frac{1}{2x+2ct} \cdot (2c) = \frac{\cancel{2}c}{\cancel{2}(x+ct)} = \frac{c}{x+ct} ,$$

$$\frac{\partial w}{\partial x} = \frac{1}{2x+2ct} \cdot (2) = \frac{\cancel{2}}{\cancel{2}(x+ct)} = \frac{1}{x+ct} ,$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c \cdot (x+ct)^{-1}\right) = -c(x+ct)^{-2} \cdot (c)$$

$$= \frac{-c^2}{(x+ct)^2} ,$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}(x+ct)^{-1} = -(x+ct)^{-2} \cdot (1) = \frac{-1}{(x+ct)^2} ;$$

then

$$\frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x+ct)^2}$$

$$= c^2 \cdot \frac{-1}{(x+ct)^2}$$

$$= c^2 \cdot \frac{\partial^2 w}{\partial x^2}$$

Section 14.4

1.) $w = x^2 + 2Y$, $x = \cos t$, $Y = \sin t$

I.) $\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_Y \cdot \frac{dY}{dt}$

$= (2x) \cdot (-\sin t) + (2) \cdot (\cos t)$

$= (2\cos t)(-\sin t) + 2\cos t$

$= 2\cos t\,(1 - \sin t)$

OR

II.) $w = x^2 + 2Y = (\cos t)^2 + 2(\sin t) \xrightarrow{D}$

$\frac{dw}{dt} = 2(\cos t) \cdot (-\sin t) + 2\cos t$

$= 2\cos t\,(1 - \sin t)$;

if $t = \pi$, then $\frac{dw}{dt} = 2\cos\pi\,(1 - \sin\pi)$

$= 2(-1)(1-0) = -2$

6.) $w = z - \sin(xY)$, $x = t$, $Y = \ln t$, $z = e^{t-1}$

I.) $\frac{dw}{dt} = w_x \cdot \frac{dx}{dt} + w_Y \cdot \frac{dY}{dt} + w_z \cdot \frac{dz}{dt}$

$= -\cos(xY) \cdot Y \cdot (1) + -\cos(xY) \cdot x \cdot \left(\frac{1}{t}\right)$
$+ (1) \cdot e^{t-1}$

$= -\cos(t\ln t) \cdot \ln t - \cos(t\ln t) \cdot \cancel{t}\left(\frac{1}{\cancel{t}}\right)$
$+ e^{t-1}$

$= -\cos(t\ln t) \cdot (\ln t + 1) + e^{t-1}$

OR

II.) $w = z - \sin(xY) = e^{t-1} - \sin(t\ln t) \xrightarrow{D}$

$\frac{dw}{dt} = e^{t-1} - \cos(t\ln t) \cdot \left[t \cdot \frac{1}{t} + (1)\ln t\right]$

$= e^{t-1} - \cos(t\ln t) \cdot [1 + \ln t]$;

if $t=1$, then $\frac{dw}{dt} = e^{0} - \cos(\overset{0}{\ln 1})\cdot[1+\overset{0}{\ln 1}]$

$= 1 - 1(1) = 0$

8.) $z = \arctan\left(\frac{x}{y}\right)$, $x = u\cos v$, $y = u\sin v$

I.) $$\frac{\partial z}{\partial u} = z_x \cdot \frac{\partial x}{\partial u} + z_y \cdot \frac{\partial y}{\partial u}$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}\cdot\cos v + \frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{-x}{y^2}\cdot\sin v$$

$$= \frac{y}{y^2+x^2}\cdot\cos v + \frac{-x}{y^2+x^2}\cdot\sin v$$

$$= \frac{u\sin v\cdot\cos v}{u^2\sin^2 v + u^2\cos^2 v} + \frac{-u\cos v\cdot\sin v}{u^2\sin^2 v + u^2\cos^2 v}$$

$= 0$ OR

II.) $z = \arctan\left(\frac{x}{y}\right) = \arctan\left(\frac{\cancel{u}\cos v}{\cancel{u}\sin v}\right) \rightarrow$

$z = \arctan(\cot v) \xrightarrow{D}$

$\frac{\partial z}{\partial u} = 0$ (since v is constant);

if $(u,v) = (1.3, \pi/6)$, then $\frac{\partial z}{\partial u} = 0$;

I.) $$\frac{\partial z}{\partial v} = z_x \cdot \frac{\partial x}{\partial v} + z_y \cdot \frac{\partial y}{\partial v}$$

$$= \frac{y}{y^2+x^2}\cdot -u\sin v + \frac{-x}{y^2+x^2}\cdot u\cos v$$

$$= \frac{u\sin v\cdot(-u\sin v)}{u^2\sin^2 v + u^2\cos^2 v} + \frac{-u\cos v\cdot(u\cos v)}{u^2\sin^2 v + u^2\cos^2 v}$$

$$= \frac{-u^2(\sin^2 v + \cos^2 v)}{u^2(\sin^2 v + \cos^2 v)} = -1$$ OR

II.) $z = \arctan\left(\frac{x}{y}\right) = \arctan(\cot v) \xrightarrow{D}$

$$\frac{\partial z}{\partial v} = \frac{1}{1+(\cot v)^2} \cdot -\csc^2 v = \frac{-\csc^2 v}{\csc^2 v} = -1 \ ;$$

if $(u,v) = (1.3, 6)$, then $\frac{\partial z}{\partial v} = -1$.

9.) $w = xy + yz + xz$, $x = u+v$, $y = u - v$, $z = uv$

I.) $\frac{\partial w}{\partial u} = w_x \cdot \frac{\partial x}{\partial u} + w_y \cdot \frac{\partial y}{\partial u} + w_z \cdot \frac{\partial z}{\partial u}$

$= (y+z)\cdot(1) + (x+z)\cdot(1) + (x+y)\cdot v$

$= (u-\cancel{v}) + uv + (u+\cancel{v}) + uv + ((u+v)+(u-v))\cdot v$

$= 2u + 2uv + 2uv = 2u + 4uv$

OR

II.) $w = xy + yz + xz$

$= (u+v)(u-v) + (u-v)(uv) + (u+v)(uv)$

$= u^2 - v^2 + u^2 v - \cancel{uv^2} + u^2 v + uv^2$

$= u^2 - v^2 + 2u^2 v \xrightarrow{D}$

$\frac{\partial w}{\partial u} = 2u + 4uv$; if $(u,v) = (\frac{1}{2}, 1)$,

then $\frac{\partial w}{\partial u} = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1) = 1 + 2 = 3$;

I.) $\frac{\partial w}{\partial v} = w_x \cdot \frac{\partial x}{\partial v} + w_y \cdot \frac{\partial y}{\partial v} + w_z \cdot \frac{\partial z}{\partial v}$

$= (y+z)(1) + (x+z)(-1) + (x+y)(u)$

$= (u-v) + uv + ((u+v)+uv)(-1) + ((u+\cancel{v})+(u-\cancel{v}))(u)$

$= \cancel{u} - v + \cancel{uv} - \cancel{u} - v - \cancel{uv} + 2u^2$

$= 2u^2 - 2v$

OR

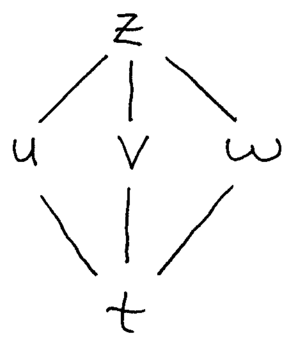
II.) $w = xy + yz + xz = u^2 - v^2 + 2u^2 v \xrightarrow{D}$

$$\frac{\partial w}{\partial v} = -2v + 2u^2 = 2u^2 - 2v \ ;$$

if $(u,v) = (\frac{1}{2}, 1)$, ~~then~~

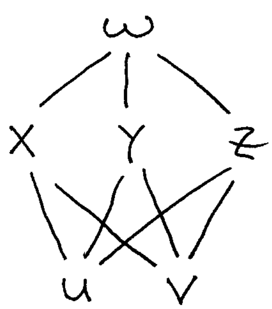
$$\frac{\partial w}{\partial v} = -2(1) + 2\left(\frac{1}{2}\right)^2 = -2 + \frac{1}{2} = -\frac{3}{2} \ .$$

14.)



$$\frac{dz}{dt} = f_u \cdot \frac{du}{dt} + f_v \cdot \frac{dv}{dt} + f_w \cdot \frac{dw}{dt}$$

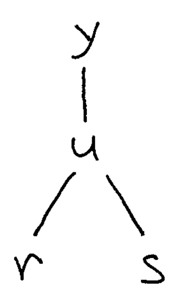
15.)



$$\frac{\partial w}{\partial u} = w_x \cdot \frac{\partial x}{\partial u} + w_y \cdot \frac{\partial y}{\partial u} + w_z \cdot \frac{\partial z}{\partial u}$$

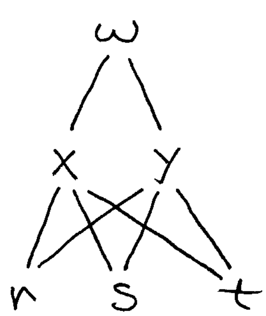
$$\frac{\partial w}{\partial v} = w_x \cdot \frac{\partial x}{\partial v} + w_y \cdot \frac{\partial y}{\partial v} + w_z \cdot \frac{\partial z}{\partial v}$$

20.)



$$\frac{\partial y}{\partial r} = \frac{dy}{du} \cdot \frac{\partial u}{\partial r}$$

24.)



$$\frac{\partial w}{\partial s} = w_x \cdot \frac{\partial x}{\partial s} + w_y \cdot \frac{\partial y}{\partial s}$$

26.) $\underbrace{xy + y^2 - 3x - 3}_{F(x,y)} = 0$;

By Theorem 8, $\dfrac{dy}{dx} = \dfrac{-F_x}{F_y} \longrightarrow$

$$\frac{dy}{dx} = \frac{-(y-3)}{x+2y} \qquad (\text{Let } (x,y) = (-1,1).) \longrightarrow$$

$$\frac{dy}{dx} = \frac{-(1-3)}{-1+2(1)} = \frac{2}{1} = 2$$

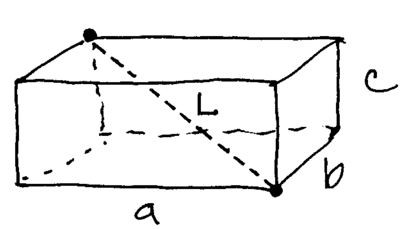
28.) $\underbrace{xe^y + \sin xy + y - \ln 2}_{F(x,y)} = 0$;

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(e^y + \cos xy \cdot y)}{xe^y + \cos xy \cdot x + 1}$$

$(\text{Let } (x,y) = (0, \ln 2).) \longrightarrow$

$$\frac{dy}{dx} = \frac{-(e^{\ln 2} + \cos 0 \cdot \ln 2)}{0 + 0 + 1} = -2 - \ln 2$$

40.)



Given $\dfrac{da}{dt} = 1$ m./sec.,

$\dfrac{db}{dt} = 1$ m./sec., and $\dfrac{dc}{dt} = -3$ m./sec.

when $a = 1$ m., $b = 2$ m., and $c = 3$ m.

volume $V = abc$; surface area $S = 2ab + 2bc + 2ac$; diagonal $L = \sqrt{a^2+b^2+c^2}$;

a.) Find $\frac{dV}{dt}$: (Use triple product rule.)

$$\frac{dV}{dt} = \frac{da}{dt}\cdot(bc) + \frac{db}{dt}(ac) + \frac{dc}{dt}(ab)$$

$$= (1)(2\cdot 3) + (1)(1\cdot 3) + (-3)(1\cdot 2)$$

$$= 6 + 3 - 6 = +3 \ m.^3/sec.$$

b.) Find $\frac{dS}{dt}$:

$$\frac{dS}{dt} = 2\left[\left(a\cdot\frac{db}{dt} + \frac{da}{dt}\cdot b\right) + \left(b\cdot\frac{dc}{dt} + \frac{db}{dt}\cdot c\right) + \left(a\cdot\frac{dc}{dt} + \frac{da}{dt}\cdot c\right)\right]$$

$$= 2\left[(1\cdot 1 + 1\cdot 2) + (2\cdot(-3) + 1\cdot 3) + (1\cdot(-3) + 1\cdot 3)\right]$$

$$= 2\left[3 + (-3) + (0)\right] = 2(0) = 0 \ m.^2/sec.$$

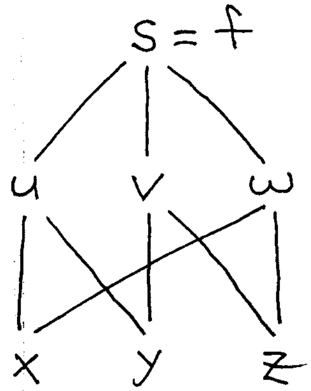
c.) Find $\frac{dL}{dt}$:

$$\frac{dL}{dt} = \frac{1}{\cancel{2}}(a^2+b^2+c^2)^{-1/2}\cdot\left[\cancel{2}a\frac{da}{dt} + \cancel{2}b\cdot\frac{db}{dt} + \cancel{2}c\cdot\frac{dc}{dt}\right]$$

$$= \frac{1}{\sqrt{14}}\left[1\cdot 1 + 2\cdot 1 + 3\cdot(-3)\right] = \frac{-6}{\sqrt{14}} \ m./sec.$$

(The diagonal is $\downarrow$.)

41.) assume function $s = f(u, v, w)$ and $u = x - y$, $v = y - z$, $w = z - x$.



Show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0 :$$

Using the chain rule $\longrightarrow$

$$\frac{\partial f}{\partial x} = s_u \cdot \frac{\partial u}{\partial x} + s_w \cdot \frac{\partial w}{\partial x}$$
$$= s_u \cdot (1) + s_w \cdot (-1) = s_u - s_w \ ;$$

$$\frac{\partial f}{\partial y} = s_u \cdot \frac{\partial u}{\partial y} + s_v \cdot \frac{\partial v}{\partial y}$$
$$= s_u \cdot (-1) + s_v \cdot (1) = s_v - s_u \ ;$$

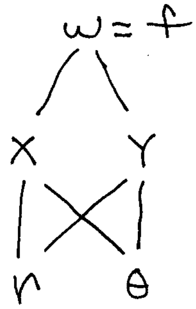
$$\frac{\partial f}{\partial z} = s_v \cdot \frac{\partial v}{\partial z} + s_w \cdot \frac{\partial w}{\partial z}$$

$$= s_v \cdot (-1) + s_w \cdot (1) = s_w - s_v \ ;$$

then

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = (\cancel{s_u} - \cancel{s_w}) + (\cancel{s_v} - \cancel{s_u}) + (\cancel{s_w} - \cancel{s_v}) = 0 .$$

42.) assume function $w = f(x, y)$ and $x = r\cos\theta$, $y = r\sin\theta$. Then



a.) $\frac{\partial w}{\partial r} = f_x \cdot \frac{\partial x}{\partial r} + f_y \cdot \frac{\partial y}{\partial r}$

$= f_x \cdot (\cos\theta) + f_y \cdot (\sin\theta)$;

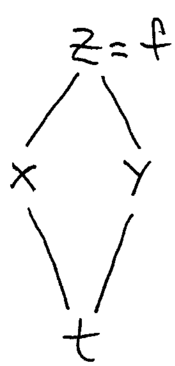
$$\frac{\partial w}{\partial \theta} = f_x \cdot \frac{\partial x}{\partial \theta} + f_y \cdot \frac{\partial y}{\partial \theta}$$

$$= f_x \cdot (-r\sin\theta) + f_y \cdot (r\cos\theta)$$

$$= r\,(-f_x \cdot \sin\theta + f_y \cdot \cos\theta) \longrightarrow$$

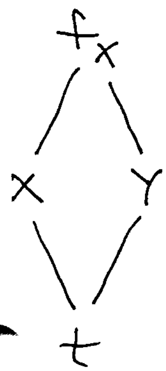
$$\frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \cdot \sin\theta + f_y \cdot \cos\theta \ .$$

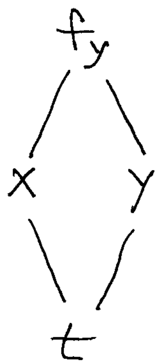
1.) a.) assume $z = f(x, y)$ and $x = e^{2t}$, $y = \sin t$. Then by the chain rule



$$\frac{dz}{dt} = f_x \cdot \frac{dx}{dt} + f_y \cdot \frac{dy}{dt}$$

$$= f_x \cdot 2e^{2t} + f_y \cdot \cos t \ ; \text{ and}$$





$$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right)$$

$$= \frac{d}{dt}\left[f_x \cdot 2e^{2t} + f_y \cdot \cos t\right]$$

$$= f_x \cdot \frac{d}{dt}(2e^{2t}) + \frac{d}{dt}(f_x) \cdot 2e^{2t}$$

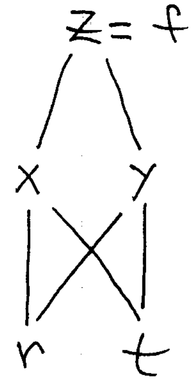
$$+ f_y \cdot \frac{d}{dt}(\cos t) + \frac{d}{dt}(f_y) \cdot \cos t$$

$$= f_x \cdot 4e^{2t} + \left[f_{xx} \cdot \frac{dx}{dt} + f_{xy} \cdot \frac{dy}{dt} \right] \cdot 2e^{2t}$$
$$+ f_y \cdot (-\sin t) + \left[f_{yx} \cdot \frac{dx}{dt} + f_{yy} \cdot \frac{dy}{dt} \right] \cdot \cos t$$

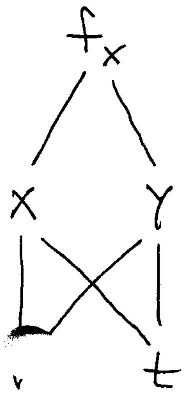
$$= f_x \cdot 4e^{2t} + \left[f_{xx} \cdot 2e^{2t} + f_{xy} \cdot \cos t \right] \cdot 2e^{2t}$$
$$- f_y \cdot \sin t + \left[f_{xy} \cdot 2e^{2t} + f_{yy} \cdot \cos t \right] \cdot \cos t$$

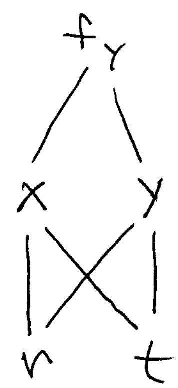
$$= f_x \cdot (4e^{2t}) - f_y \cdot (\sin t)$$
$$+ f_{xx} \cdot (4e^{4t}) + f_{yy} \cdot (\cos^2 t)$$
$$+ f_{xy} \cdot (4e^{2t} \cos t) \quad .$$

1.) 6.) assume $z = f(x, y)$ and $x = rt^2$, $y = r^3 - t$.
Then by chain rule



i.) $$\frac{\partial z}{\partial t} = f_x \cdot \frac{\partial x}{\partial t} + f_y \cdot \frac{\partial y}{\partial t}$$
$$= f_x \cdot (2rt) + f_y \cdot (-1)$$
$$= f_x \cdot (2rt) - f_y \quad ; \text{ then}$$





$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial t} \right]$$
$$= \frac{\partial}{\partial t}\left[f_x \cdot (2rt) - f_y \right]$$
$$= f_x \cdot \frac{\partial}{\partial t}(2rt) + \frac{\partial}{\partial t}(f_x) \cdot (2rt)$$
$$- \frac{\partial}{\partial t}(f_y)$$

$$= f_X \cdot 2r + \left[f_{XX} \cdot \frac{\partial X}{\partial t} + f_{XY} \cdot \frac{\partial Y}{\partial t} \right] \cdot (2rt)$$
$$- \left[f_{YX} \cdot \frac{\partial X}{\partial t} + f_{YY} \cdot \frac{\partial Y}{\partial t} \right]$$
$$= f_X \cdot 2r + f_{XX} \cdot (2rt)(2rt) + f_{XY} \cdot (-1)(2rt)$$
$$- f_{XY} \cdot (2rt) - f_{YY} \cdot (-1)$$
$$= f_X \cdot (2r) + f_{XX} \cdot (4r^2 t^2)$$
$$- f_{XY} \cdot (4rt) + f_{YY}$$

ii.) $\frac{\partial z}{\partial r} = f_X \cdot \frac{\partial X}{\partial r} + f_Y \cdot \frac{\partial Y}{\partial r}$

$$= f_X \cdot t^2 + f_Y \cdot 3r^2$$; then

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial z}{\partial r} \right] = \frac{\partial}{\partial r}\left[f_X \cdot t^2 + f_Y \cdot 3r^2 \right]$$
$$= \frac{\partial}{\partial r}\left[f_X \right] \cdot t^2 + f_Y \cdot \frac{\partial}{\partial r}(3r^2) + \frac{\partial}{\partial r}(f_Y) \cdot 3r^2$$
$$= \left[f_{XX} \cdot \frac{\partial X}{\partial r} + f_{XY} \cdot \frac{\partial Y}{\partial r} \right] \cdot t^2$$
$$+ f_Y \cdot 6r + \left[f_{YX} \cdot \frac{\partial X}{\partial r} + f_{YY} \cdot \frac{\partial Y}{\partial r} \right] \cdot 3r^2$$
$$= f_{XX} \cdot (t^2)(t^2) + f_{XY} (3r^2)(t^2)$$
$$+ f_Y \cdot 6r + f_{XY} \cdot (t^2)(3r^2) + f_{YY} \cdot (3r^2)(3r^2)$$
$$= f_{XX} \cdot (t^4) + f_Y \cdot (6r) + f_{YY} \cdot (9r^4)$$
$$+ f_{XY} \cdot (6r^2 t^2)$$