

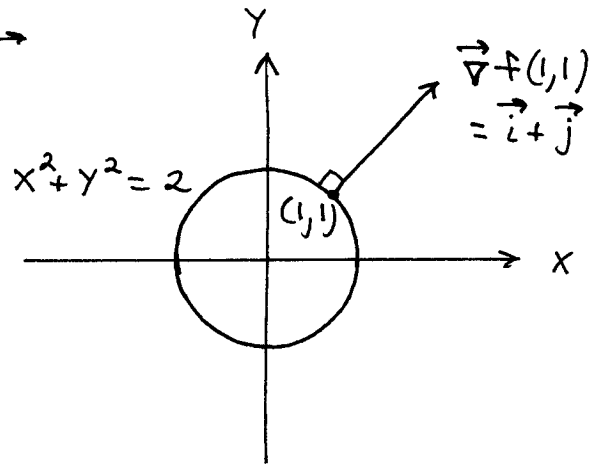
Section 14.5

- 2.) $f(x, y) = \ln(x^2 + y^2)$ and $P = (1, 1)$;
 $f(1, 1) = \ln 2$ then level curve at $(1, 1)$
 is $\ln 2 = \ln(x^2 + y^2) \rightarrow$

$$\underline{x^2 + y^2 = 2} ; \text{ and } f_x = \frac{2x}{x^2 + y^2} \text{ and}$$

$$f_y = \frac{2y}{x^2 + y^2} \text{ so gradient vector is}$$

$$\begin{aligned} \vec{\nabla} f(1, 1) &= f_x(1, 1) \vec{i} + f_y(1, 1) \vec{j} \\ &= (1) \vec{i} + (1) \vec{j} = \vec{i} + \vec{j} \end{aligned}$$

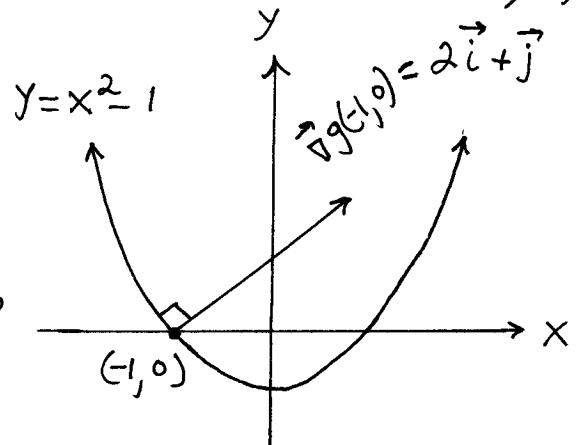


- 3.) $g(x, y) = y - x^2$ and $P = (-1, 0)$;
 $g(-1, 0) = 0 - (-1)^2 = -1$ then level curve at $(-1, 0)$
 is $-1 = y - x^2 \rightarrow$

$$\underline{y = x^2 - 1} ; \text{ and}$$

$$g_x = -2x, \quad g_y = 1 \text{ so gradient vector is}$$

$$\begin{aligned} \vec{\nabla} g(-1, 0) &= g_x(-1, 0) \vec{i} + g_y(-1, 0) \vec{j} \\ &= (2) \vec{i} + (1) \vec{j} = 2\vec{i} + \vec{j} \end{aligned}$$



- 5.) $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x$
 and $P = (1, 1, 1)$; then

$$f_x = 2x + z \cdot \frac{1}{x}, \quad f_y = 2y, \quad f_z = -4z + \ln x$$

so that the gradient vector is

$$\vec{\nabla} f(1, 1, 1) = f_x(1, 1, 1) \vec{i} + f_y(1, 1, 1) \vec{j} + f_z(1, 1, 1) \vec{k}$$

$$= 3\vec{i} + 2\vec{j} - 4\vec{k}$$

8.) $f(x, y, z) = e^{x+y} \cos z + (y+1) \arcsin x$

and $P = (0, 0, \pi/6)$; then

$$f_x = e^{x+y} \cos z + (y+1) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$f_y = e^{x+y} \cos z + \arcsin x,$$

$$f_z = -e^{x+y} \cdot \sin z \quad \text{so that the}$$

gradient vector is

$$\vec{\nabla} f(0, 0, \pi/6) = f_x(0, 0, \pi/6) \vec{i} + f_y(0, 0, \pi/6) \vec{j} + f_z(0, 0, \pi/6) \vec{k}$$

$$= \left(\frac{\sqrt{3}}{2} + 1\right) \vec{i} + \frac{\sqrt{3}}{2} \vec{j} - \frac{1}{2} \vec{k}$$

10.) $f(x, y) = 2x^2 + y^2$, $P = (-1, 1)$, and

$\vec{A} = 3\vec{i} - 4\vec{j}$; then direction vector

$$\vec{u} = \frac{1}{|\vec{A}|} \vec{A} = \frac{1}{5} (3\vec{i} - 4\vec{j}) = \frac{3}{5} \vec{i} - \frac{4}{5} \vec{j}; \text{ and}$$

$f_x = 4x$, $f_y = 2y$ so gradient vector

$$\vec{\nabla} f(-1, 1) = f_x(-1, 1) \vec{i} + f_y(-1, 1) \vec{j}; \text{ then}$$

$$D_{\vec{u}} f(-1, 1) = \vec{\nabla} f(-1, 1) \cdot \vec{u}$$

$$= (-4\vec{i} + 2\vec{j}) \cdot \left(\frac{3}{5} \vec{i} - \frac{4}{5} \vec{j}\right)$$

$$= \frac{-12}{5} - \frac{8}{5} = \boxed{-4}$$

13.) $f(x, y, z) = xy + yz + xz$, $P = (1, -1, 2)$, and

$\vec{A} = 3\vec{i} + 6\vec{j} - 2\vec{k}$; then direction vector

$$\vec{u} = \frac{1}{|\vec{A}|} \vec{A} = \frac{1}{7} (3\vec{i} + 6\vec{j} - 2\vec{k}) = \frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k} ;$$

and $f_x = y + z$, $f_y = x + z$, $f_z = x + y$ so gradient vector is

$$\begin{aligned} \vec{\nabla} f(1, -1, 2) &= f_x(1, -1, 2)\vec{i} + f_y(1, -1, 2)\vec{j} + f_z(1, -1, 2)\vec{k} \\ &= (1)\vec{i} + (3)\vec{j} + (0)\vec{k} = \vec{i} + 3\vec{j} ; \text{ then} \end{aligned}$$

$$\begin{aligned} D_{\vec{u}} f(1, -1, 2) &= \vec{\nabla} f(1, -1, 2) \cdot \vec{u} \\ &= (\vec{i} + 3\vec{j}) \cdot \left(\frac{3}{7}\vec{i} + \frac{6}{7}\vec{j} - \frac{2}{7}\vec{k} \right) \\ &= \frac{3}{7} + \frac{18}{7} = \textcircled{3} . \end{aligned}$$

14.) $f(x, y, z) = x^2 + 2y^2 - 3z^2$, $P = (1, 1, 1)$, and

$\vec{A} = \vec{i} + \vec{j} + \vec{k}$; then direction vector

$$\vec{u} = \frac{1}{|\vec{A}|} \vec{A} = \frac{1}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k}) = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k} ;$$

and $f_x = 2x$, $f_y = 4y$, $f_z = -6z$ so gradient vector is

$$\begin{aligned} \vec{\nabla} f(1, 1, 1) &= f_x(1, 1, 1)\vec{i} + f_y(1, 1, 1)\vec{j} + f_z(1, 1, 1)\vec{k} \\ &= (2)\vec{i} + (4)\vec{j} + (-6)\vec{k} = 2\vec{i} + 4\vec{j} - 6\vec{k} ; \end{aligned}$$

then $D_{\vec{u}} f(1, 1, 1) = \vec{\nabla} f(1, 1, 1) \cdot \vec{u}$

$$\begin{aligned} &= (2\vec{i} + 4\vec{j} - 6\vec{k}) \cdot \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k} \right) \\ &= \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{-6}{\sqrt{3}} = \textcircled{0} . \end{aligned}$$

17.) $f(x,y) = x^2 + xy + y^2$ and $P = (-1, 1)$;
 $f_x = 2x + y$, $f_y = x + 2y$ so gradient
 vector is

$$\begin{aligned}\vec{\nabla} f(-1, 1) &= f_x(-1, 1) \vec{i} + f_y(-1, 1) \vec{j} \\ &= (-1) \vec{i} + (1) \vec{j} = -\vec{i} + \vec{j} ;\end{aligned}$$

a.) The direction of maximum increase
 is in the same direction as $\vec{\nabla} f(-1, 1)$, i.e.,
 $\vec{u} = \frac{1}{|\vec{\nabla} f(-1, 1)|} \vec{\nabla} f(-1, 1) = \frac{1}{\sqrt{2}} (-\vec{i} + \vec{j}) = \frac{-1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}$;

the dir. der. in this direction is

$$D_{\vec{u}} f(-1, 1) = |\vec{\nabla} f(-1, 1)| = \sqrt{2} .$$

b.) The direction of maximum decrease
 is in the opposite direction of $\vec{\nabla} f(-1, 1)$,
 i.e., $\vec{u} = \frac{1}{\sqrt{2}} \vec{i} - \frac{1}{\sqrt{2}} \vec{j}$; the dir. der.

in this direction is

$$D_{\vec{u}} f(-1, 1) = -|\vec{\nabla} f(-1, 1)| = -\sqrt{2}$$

20.) $g(x, y, z) = xe^y + z^2$ and $P = (1, \ln 2, \frac{1}{2})$;
 $g_x = e^y$, $g_y = xe^y$, $g_z = 2z$ so gradient
 vector is

$$\begin{aligned}\vec{\nabla} g(1, \ln 2, \frac{1}{2}) &= g_x(1, \ln 2, \frac{1}{2}) \vec{i} + g_y(1, \ln 2, \frac{1}{2}) \vec{j} + g_z(1, \ln 2, \frac{1}{2}) \vec{k} \\ &= (2) \vec{i} + (2) \vec{j} + (1) \vec{k} = 2\vec{i} + 2\vec{j} + \vec{k}\end{aligned}$$

a.) The direction of maximum increase
 is in the same direction as $\vec{\nabla} g(1, \ln 2, \frac{1}{2})$;

$$\text{i.e., } \vec{u} = \frac{1}{|\vec{\nabla} g(1, \ln 2, \frac{1}{2})|} \vec{\nabla} g(1, \ln 2, \frac{1}{2})$$

$$= \frac{1}{3} (2\vec{i} + 2\vec{j} + \vec{k}) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k} ;$$

the dir. der. in this direction is

$$D_{\vec{u}} g(1, \ln 2, \frac{1}{2}) = |\vec{\nabla} g(1, \ln 2, \frac{1}{2})| = \textcircled{3} .$$

b.) The direction of maximum decrease is in the opposite direction of $\vec{\nabla} g(1, \ln 2, \frac{1}{2})$,
i.e., $\vec{u} = -\frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$; the dir. der.

in this direction is

$$D_{\vec{u}} g(1, \ln 2, \frac{1}{2}) = -|\vec{\nabla} g(1, \ln 2, \frac{1}{2})| = \textcircled{-3} .$$

27.) Let $f(x, y) = xy + y^2$ and $P = (3, 2)$.

For what $\vec{u}$ is $D_{\vec{u}} f(3, 2) = 0$?

$$D_{\vec{u}} f(3, 2) = \vec{\nabla} f(3, 2) \cdot \vec{u} = 0 \Rightarrow$$

$\vec{u}$ is parallel to $\vec{T} = f_y(3, 2)\vec{i} - f_x(3, 2)\vec{j}$;
 $f_x = y$ and $f_y = x + 2y$ so

$$\vec{T} = 7\vec{i} - 2\vec{j} ; \text{ then}$$

$$\vec{u} = \frac{1}{|\vec{T}|} \vec{T} = \frac{1}{\sqrt{53}} (7\vec{i} - 2\vec{j}) = \frac{7}{\sqrt{53}} \vec{i} - \frac{2}{\sqrt{53}} \vec{j}$$

$$\text{or } \vec{u} = \frac{-1}{|\vec{T}|} \vec{T} = \frac{-7}{\sqrt{53}} \vec{i} + \frac{2}{\sqrt{53}} \vec{j}$$

31.) If $\vec{A} = \vec{i} + \vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|} \vec{A} = \frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}$

and $D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = 2\sqrt{2} \Rightarrow$

$$(f_x(1,2) \vec{i} + f_y(1,2) \vec{j}) \cdot \left(\frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j} \right) = 2\sqrt{2} \Rightarrow$$

$$f_x(1,2) \cdot \frac{1}{\sqrt{2}} + f_y(1,2) \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2} \Rightarrow$$

$$\boxed{f_x(1,2) + f_y(1,2) = 4} ;$$

if $\vec{A} = -2\vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|} \vec{A} = \frac{1}{2}(-2\vec{j}) = -\vec{j}$

and $D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = -3 \Rightarrow$

$$(f_x(1,2) \vec{i} + f_y(1,2) \vec{j}) \cdot (-\vec{j}) = -3 \Rightarrow$$

$$-f_y(1,2) = -3 \Rightarrow \boxed{f_y(1,2) = 3} ; \text{ then}$$

$$f_x(1,2) + (3) = 4 \Rightarrow \boxed{f_x(1,2) = 1} ;$$

now the gradient vector is

$$\underline{\vec{\nabla} f(1,2)} = f_x(1,2) \vec{i} + f_y(1,2) \vec{j} = \underline{\vec{i} + 3\vec{j}} ;$$

if $\vec{A} = -\vec{i} - 2\vec{j}$, then $\vec{u} = \frac{1}{|\vec{A}|} \vec{A} = \frac{1}{\sqrt{5}} (-\vec{i} - 2\vec{j})$

$$\Rightarrow \vec{u} = -\frac{1}{\sqrt{5}} \vec{i} - \frac{2}{\sqrt{5}} \vec{j} ; \text{ then}$$

$$D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u}$$

$$= (\vec{i} + 3\vec{j}) \cdot \left(-\frac{1}{\sqrt{5}} \vec{i} - \frac{2}{\sqrt{5}} \vec{j} \right)$$

$$= -\frac{1}{\sqrt{5}} - \frac{6}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$

Let $P = (a, b, c)$ and $w = f(x, y, z)$.

32.) a.) If $\vec{V} = \vec{i} + \vec{j} - \vec{k}$, then

$$\vec{u} = \frac{1}{|\vec{V}|} \vec{V} = \frac{1}{\sqrt{3}} (\vec{i} + \vec{j} - \vec{k}) = \frac{1}{\sqrt{3}} \vec{i} + \frac{1}{\sqrt{3}} \vec{j} - \frac{1}{\sqrt{3}} \vec{k} ;$$

$$\text{since } D_{\vec{u}} f(a, b, c) = \vec{\nabla} f(a, b, c) \cdot \vec{u} = 2\sqrt{3} \Rightarrow$$

$$(f_x(a, b, c) \vec{i} + f_y(a, b, c) \vec{j} + f_z(a, b, c) \vec{k}) \cdot \left(\frac{1}{\sqrt{3}} \vec{i} + \frac{1}{\sqrt{3}} \vec{j} - \frac{1}{\sqrt{3}} \vec{k} \right) = 2\sqrt{3} \Rightarrow$$

$$f_x(a, b, c) \cdot \frac{1}{\sqrt{3}} + f_y(a, b, c) \cdot \frac{1}{\sqrt{3}} - f_z(a, b, c) \cdot \frac{1}{\sqrt{3}} = 2\sqrt{3} \Rightarrow$$

$$(1) \quad \boxed{f_x(a, b, c) + f_y(a, b, c) + f_z(a, b, c) = 6} ;$$

since $D_{\vec{u}} f(a, b, c)$ is the largest value of a directional derivative, $\vec{u}$ and $\vec{\nabla} f(a, b, c)$ point in the same direction, i.e.,

$$\boxed{\vec{\nabla} f(a, b, c) = k \vec{u}}$$

for some constant k . Then

$$f_x(a, b, c) \vec{i} + f_y(a, b, c) \vec{j} + f_z(a, b, c) \vec{k} = \frac{k}{\sqrt{3}} \vec{i} + \frac{k}{\sqrt{3}} \vec{j} - \frac{k}{\sqrt{3}} \vec{k} \Rightarrow$$

$$f_x(a, b, c) = \frac{k}{\sqrt{3}}, \quad f_y(a, b, c) = \frac{k}{\sqrt{3}}, \quad f_z(a, b, c) = -\frac{k}{\sqrt{3}} \Rightarrow$$

$$(\text{Use (1) now.}) \quad \frac{k}{\sqrt{3}} + \frac{k}{\sqrt{3}} + \frac{-k}{\sqrt{3}} = 6 \Rightarrow$$

$$\frac{k}{\sqrt{3}} = 6 \Rightarrow k = 6\sqrt{3} \Rightarrow$$

$$\boxed{\vec{\nabla} f(a, b, c) = 6\vec{i} + 6\vec{j} - 6\vec{k}} .$$