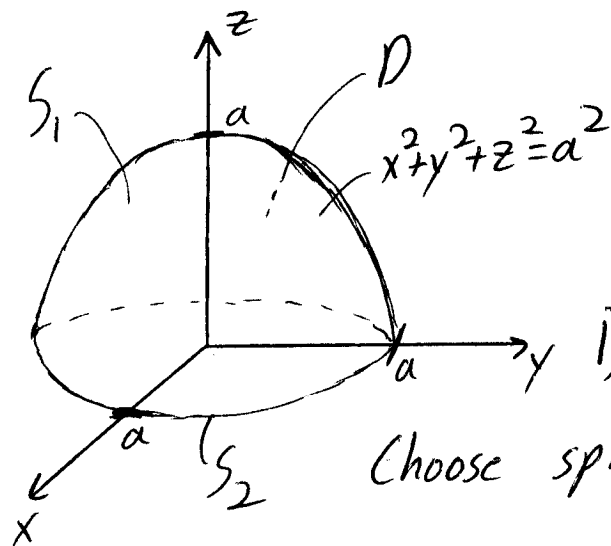


Divergence Theorem Example

Ex Verify the Divergence Theorem for $\vec{F}(x,y,z) = (z^2+2)\vec{k}$ for D : top half of sphere $x^2+y^2+z^2=a^2$ with base the disk $x^2+y^2 \leq a^2$ in xy -plane.



Divergence Thm

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iiint_D \operatorname{div} \vec{F} \, dV \quad (1)$$

$$\vec{F} = (z^2+2)\vec{k} \Rightarrow \operatorname{div} \vec{F} = 0+0+2z = 2z$$

$$1) \iiint_D \operatorname{div} \vec{F} \, dV = \iiint_D 2z \, dV$$

Choose spherical coords:

$$D: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/2 \\ 0 \leq \rho \leq a \end{cases}$$

$$\Rightarrow \iiint_D 2z \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a 2\rho \cos \phi \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 2\rho^3 \cos \phi \sin \phi \, d\rho \, d\phi \, d\theta = 2 \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{4} \rho^4 \Big|_{\rho=0}^{\rho=a} \cos \phi \sin \phi \, d\phi \, d\theta$$

$$= \frac{1}{2} a^4 \int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta = \frac{1}{2} a^4 \int_0^{2\pi} \frac{1}{2} \sin^2 \phi \Big|_{\phi=0}^{\phi=\pi/2} d\theta$$

$$= \frac{1}{4} a^4 \int_0^{2\pi} d\theta = \frac{1}{2} a^4 \pi$$

$$2) \iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iint_{S_1} \vec{F} \cdot \vec{n} \, d\sigma + \iint_{S_2} \vec{F} \cdot \vec{n} \, d\sigma \quad \text{where } \begin{matrix} S_1: \frac{1}{2} \text{ sphere} \\ S_2: \text{ Disk} \end{matrix}$$

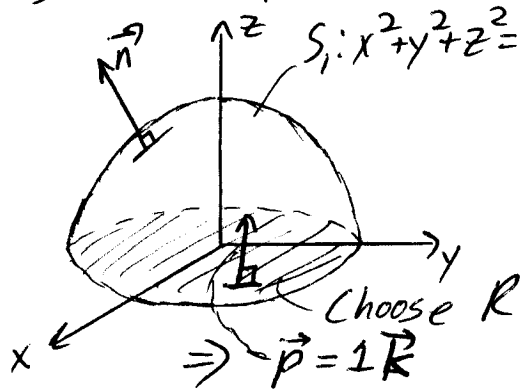
$$S_1: x^2+y^2+z^2=a^2$$

Let $f(x,y,z) = x^2+y^2+z^2$
(Implicit Surface)

$$\Rightarrow \vec{\nabla} f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\vec{\nabla} f| = \sqrt{4x^2+4y^2+4z^2} = \sqrt{4a^2} = 2a$$

$$\Rightarrow \vec{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{x}{a}\vec{i} + \frac{y}{a}\vec{j} + \frac{z}{a}\vec{k}$$



'Exchange Rate': $\frac{|\vec{\nabla}f|}{|\vec{\nabla}f \cdot \vec{p}|} = \frac{|\vec{\nabla}f|}{|\vec{\nabla}f \cdot \vec{k}|} = \frac{2a}{|2z|} = \frac{a}{z}$

Note: $-\vec{\nabla}f \cdot \vec{k} = (2x\vec{i} + 2y\vec{j} + 2z\vec{k}) \cdot (0\vec{i} + 0\vec{j} + 1\vec{k}) = 0 + 0 + 2z = 2z$
 $-\vec{F} \cdot \vec{n} = (0\vec{i} + 0\vec{j} + (z^2+2)\vec{k}) \cdot (\frac{x}{a}\vec{i} + \frac{y}{a}\vec{j} + \frac{z}{a}\vec{k}) = 0 + 0 + (z^2+2)\frac{z}{a} = (z^2+2)\frac{z}{a}$

$$\iint_{S_1} \vec{F} \cdot \vec{n} d\sigma = \iint_{S_1} (z^2+2) \cdot \frac{z}{a} d\sigma$$

$$= \iint_R (z^2+2) \frac{z}{a} \cdot \frac{dA}{z} = \iint_R (z^2+2) dA$$

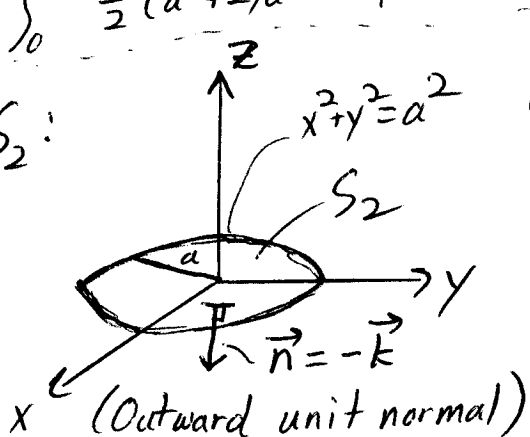
$$= \int_0^{2\pi} \int_0^a (a^2 - r^2 + 2) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^a (a^2+2)r - r^3 dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}(a^2+2)r^2 - \frac{1}{4}r^4 \right]_{r=0}^{r=a} d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}(a^2+2)a^2 - \frac{1}{4}a^4 \right] d\theta = \left(\frac{1}{4}a^4 + a^2 \right) \int_0^{2\pi} d\theta = \frac{1}{2}\pi a^4 + 2\pi a^2$$

S_2 :



Note: $\vec{F} \cdot \vec{n} = (0\vec{i} + 0\vec{j} + (z^2+2)\vec{k}) \cdot (0\vec{i} + 0\vec{j} - 1\vec{k}) = -(z^2+2)$

$$\iint_{S_2} \vec{F} \cdot \vec{n} d\sigma = \iint_{S_2} -(z^2+2) d\sigma$$

(Since S_2 is already 'flat', this is just a double integral)

$$\iint_{S_2} -(z^2+2) d\sigma = \iint_{S_2} -(0+2) d\sigma = -2 \iint_{S_2} d\sigma = -2(\text{Area of } S_2)$$

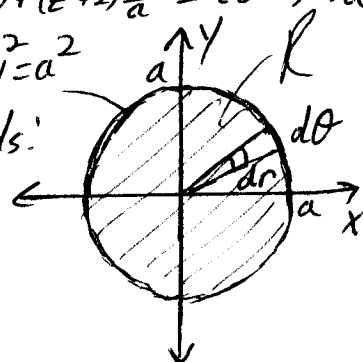
$$= -2\pi a^2$$

Hence, $\iint_S \vec{F} \cdot \vec{n} d\sigma = \iint_{S_1} \vec{F} \cdot \vec{n} d\sigma + \iint_{S_2} \vec{F} \cdot \vec{n} d\sigma$

$$= \frac{1}{2}\pi a^4 + 2\pi a^2 - 2\pi a^2 = \frac{1}{2}\pi a^4 \quad \text{Stoke's}$$

Choose Polar coords:

$$R: \begin{aligned} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq a \end{aligned}$$



Note: $x^2 + y^2 + z^2 = a^2 \Rightarrow z^2 = a^2 - x^2 - y^2$
 $\Rightarrow z^2 = a^2 - r^2$