

Section 15.6

$$\begin{aligned}
 1.) & \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz \cdot r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 \left(z \Big|_{z=r}^{z=\sqrt{2-r^2}} \right) r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 (\sqrt{2-r^2} - r) r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 (r\sqrt{2-r^2} - r^2) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left(-\frac{1}{3}(2-r^2)^{3/2} - \frac{1}{3}r^3 \right) \Big|_{r=0}^{r=1} d\theta \\
 &= \int_0^{2\pi} \left[\left(-\frac{1}{3} - \frac{1}{3} \right) - \left(-\frac{1}{3} \cdot 2^{3/2} - 0 \right) \right] d\theta \\
 &= \int_0^{2\pi} \left(-\frac{2}{3} + \frac{1}{3} 2^{3/2} \right) d\theta = \left(-\frac{2}{3} + \frac{1}{3} \cdot 2 \cdot \sqrt{2} \right) \theta \Big|_0^{2\pi} \\
 &= \left(-\frac{2}{3} + \frac{2}{3} \sqrt{2} \right) (2\pi - 0) = \frac{4\pi}{3} (\sqrt{2} - 1)
 \end{aligned}$$

$$\begin{aligned}
 6.) & \int_0^{2\pi} \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} (r^2 \sin^2 \theta + z^2) \, dz \cdot r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 \left(r^2 \sin^2 \theta \cdot z + \frac{1}{3} z^3 \right) \Big|_{z=-\frac{1}{2}}^{z=\frac{1}{2}} \cdot r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 \left[\left(r^2 \sin^2 \theta \cdot \frac{1}{2} + \frac{1}{3} \left(\frac{1}{8} \right) \right) \right. \\
 &\quad \left. - \left(r^2 \sin^2 \theta \left(-\frac{1}{2} \right) + \frac{1}{3} \left(-\frac{1}{8} \right) \right) \right] r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 \left(\frac{1}{2} + r^2 \sin^2 \theta \right) \cdot r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 \left(\frac{1}{2} r + r^3 \sin^2 \theta \right) \, dr \, d\theta
 \end{aligned}$$

$$= \int_0^{2\pi} \left(\frac{1}{24} r^2 + \frac{1}{4} r^4 \sin^2 \theta \right) \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{24} + \frac{1}{4} \sin^2 \theta \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{24} + \frac{1}{4} \cdot \frac{1}{2} (1 - \cos 2\theta) \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{24} + \frac{1}{8} - \frac{1}{8} \cos 2\theta \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{6} - \frac{1}{8} \cos 2\theta \right) d\theta$$

$$= \left(\frac{1}{6} \theta - \frac{1}{8} \cdot \frac{1}{2} \sin 2\theta \right) \Big|_0^{2\pi}$$

$$= \left(\frac{2\pi}{6} - \frac{1}{16} \sin 4\pi \right) - \left(0 - \frac{1}{16} \sin 0 \right) = \frac{\pi}{3}$$

$$8.) \int_{-1}^1 \int_0^{2\pi} \int_0^{1+\cos\theta} 4r \, dr \, d\theta \, dz$$

$$= \int_{-1}^1 \int_0^{2\pi} \left(2r^2 \Big|_{r=0}^{r=1+\cos\theta} \right) d\theta \, dz$$

$$= \int_{-1}^1 \int_0^{2\pi} 2(1+\cos\theta)^2 d\theta \, dz$$

$$= \int_{-1}^1 \int_0^{2\pi} 2(1+2\cos\theta+\cos^2\theta) d\theta \, dz$$

$$= \int_{-1}^1 \int_0^{2\pi} 2 \left(1+2\cos\theta + \frac{1}{2} (1+\cos 2\theta) \right) d\theta \, dz$$

$$= \int_{-1}^1 \int_0^{2\pi} (3+4\cos\theta+\cos 2\theta) d\theta \, dz$$

$$= \int_{-1}^1 \left(3\theta + 4\sin\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{\theta=0}^{\theta=2\pi} dz$$

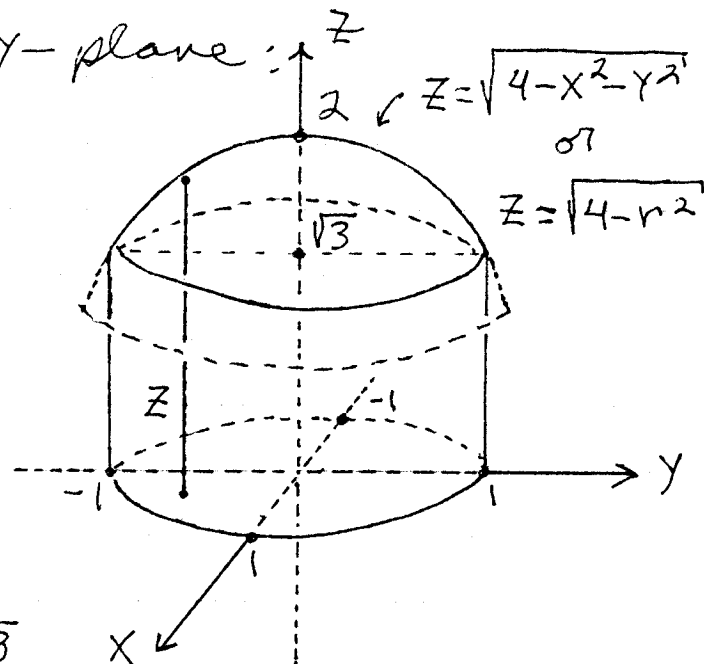
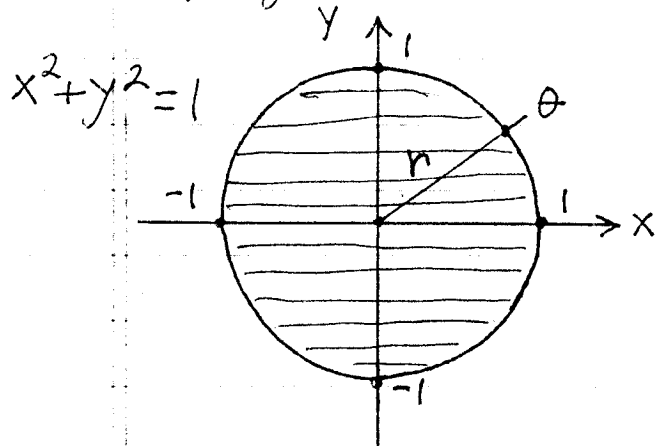
$$= \int_{-1}^1 \left[(6\pi + 4\sin 2\pi + \frac{1}{2} \sin 4\pi) - (0 + 4\sin 0 + \frac{1}{2} \sin 0) \right] dz$$

$$= \int_{-1}^1 6\pi \, dz = 6\pi \cdot z \Big|_{-1}^1 = 6\pi(1 - (-1))$$

$$= 12\pi$$

$$\begin{aligned}
 10.) \quad & \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r \sin \theta + 1) r \, d\theta \, dz \, dr \\
 &= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r^2 \sin \theta + r) \, d\theta \, dz \, dr \\
 &= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} (-r^2 \cos \theta + r\theta) \Big|_{\theta=0}^{\theta=2\pi} \, dz \, dr \\
 &= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} [(-r^2 \cancel{\cos 2\pi} + r \cdot 2\pi) - (-r^2 \cancel{\cos 0} + r \cdot 0)] \, dz \, dr \\
 &= \int_0^2 \int_{r-2}^{\sqrt{4-r^2}} (-\cancel{r^2} + 2\pi r + \cancel{r^2}) \, dz \, dr \\
 &= \int_0^2 \left(2\pi r \cdot z \Big|_{z=r-2}^{z=\sqrt{4-r^2}} \right) \, dr \\
 &= \int_0^2 2\pi r (\sqrt{4-r^2} - (r-2)) \, dr \\
 &= 2\pi \int_0^2 (r \cdot \sqrt{4-r^2} - r^2 + 2r) \, dr \\
 &= 2\pi \cdot \left(-\frac{1}{3} (4-r^2)^{3/2} - \frac{1}{3} r^3 + r^2 \right) \Big|_0^2 \\
 &= 2\pi \left[\left(0 - \frac{8}{3} + 4 \right) - \left(-\frac{1}{3} (4)^{3/2} - 0 + 0 \right) \right] \\
 &= 2\pi \left[\cancel{-\frac{8}{3}} + \frac{12}{3} + \cancel{\frac{8}{3}} \right] = 8\pi
 \end{aligned}$$

11.) projection onto xy -plane:



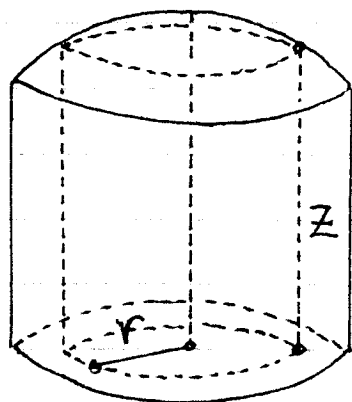
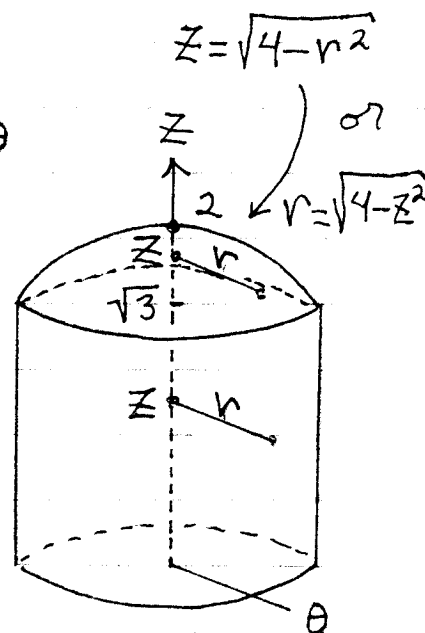
$$x^2 + y^2 + z^2 = 4, x^2 + y^2 = 1 \rightarrow$$

$$1 + z^2 = 4 \rightarrow z^2 = 3 \rightarrow z = \sqrt{3}$$

a.) $Vol = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$

b.) $Vol = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 r \, dr \, dz \, d\theta$

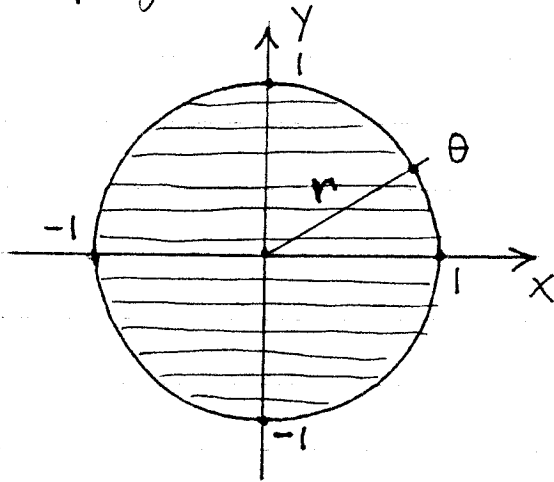
$$+ \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta$$



c.)

$$Vol = \int_0^1 \int_0^{\sqrt{4-r^2}} \int_0^{2\pi} r \, d\theta \, dz \, dr$$

12.) projection onto xy -plane:



$$z = \sqrt{x^2 + y^2} \text{ and}$$

$$z = 2 - x^2 - y^2 \rightarrow$$

$$x^2 + y^2 = 2 - z \rightarrow$$

$$z = \sqrt{2 - z} \rightarrow$$

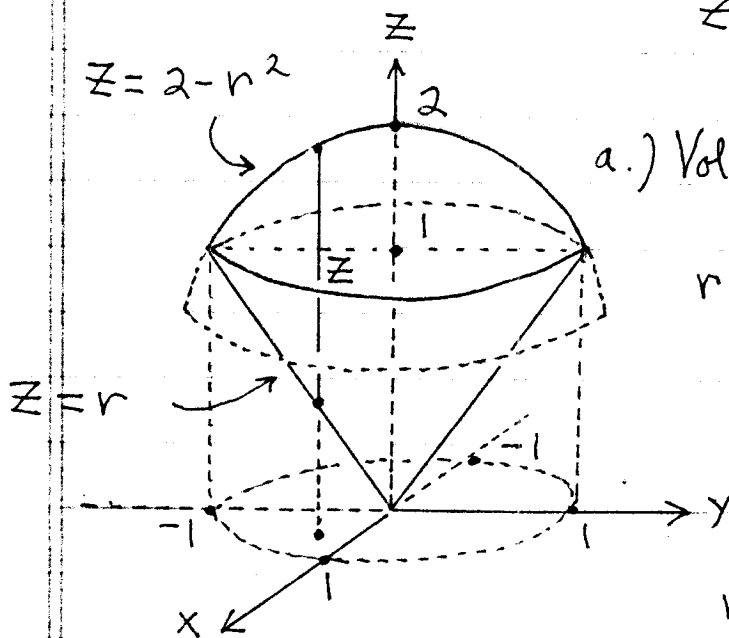
$$z^2 = 2 - z \rightarrow$$

$$z^2 + z - 2 = 0 \rightarrow$$

$$(z - 1)(z + 2) = 0 \rightarrow$$

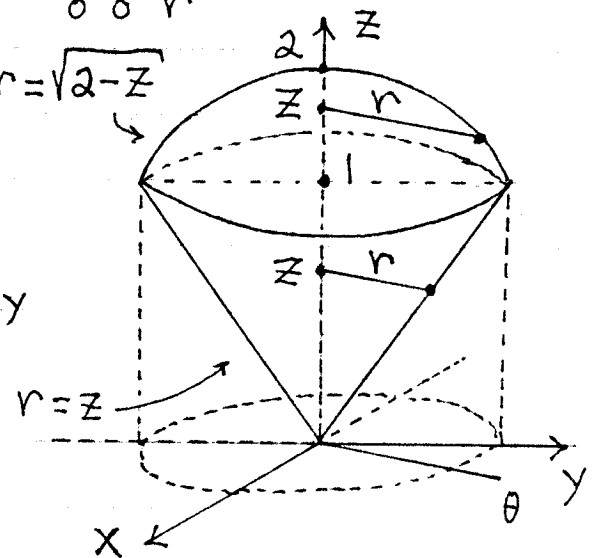
$$\downarrow \quad \downarrow$$

$$z = 1 \quad z = -2 \rightarrow x^2 + y^2 = 1$$



$$a.) Vol = \int_0^{2\pi} \int_0^1 \int_{2-r^2}^1 r \, dz \, dr \, d\theta$$

$$r = \sqrt{2 - z}$$



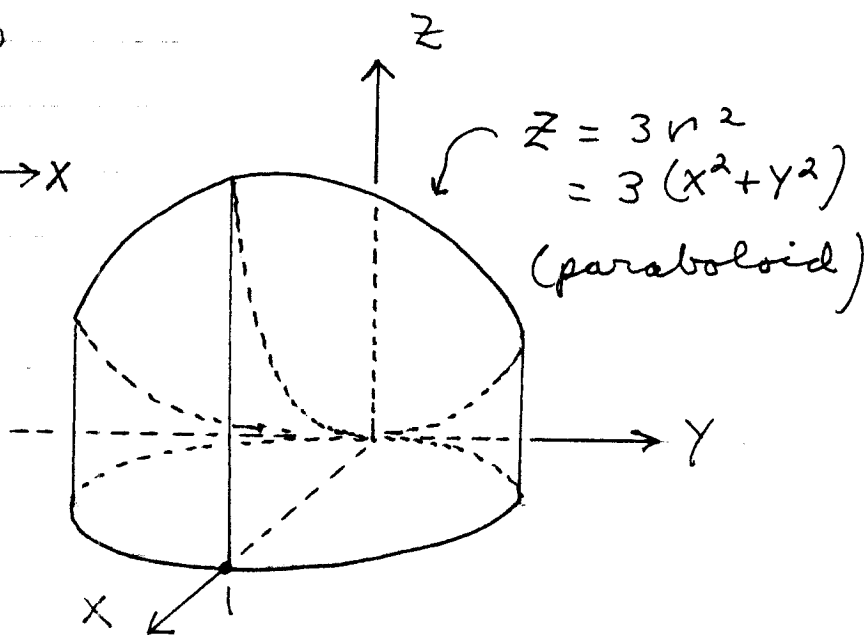
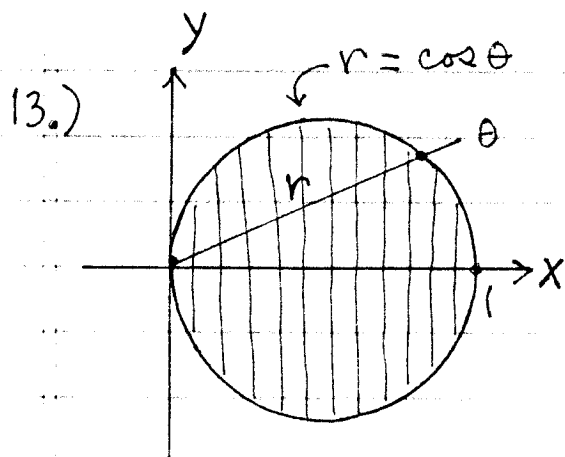
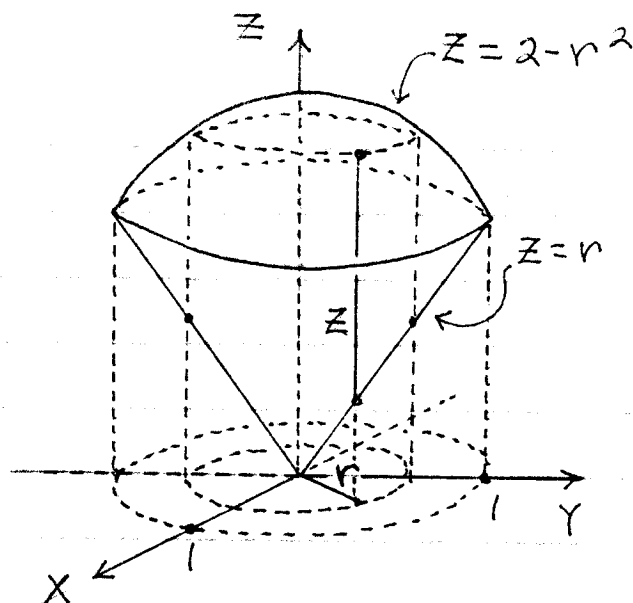
b.)

$$Vol = \int_0^{2\pi} \int_0^1 \int_0^z r \, dr \, dz \, d\theta$$

$$+ \int_0^{2\pi} \int_1^2 \int_0^{\sqrt{2-z}} r \, dr \, dz \, d\theta$$

c.)

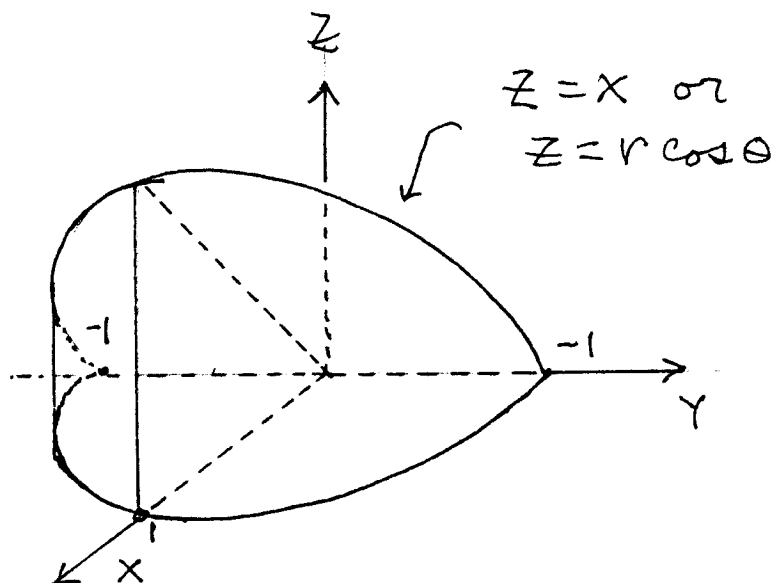
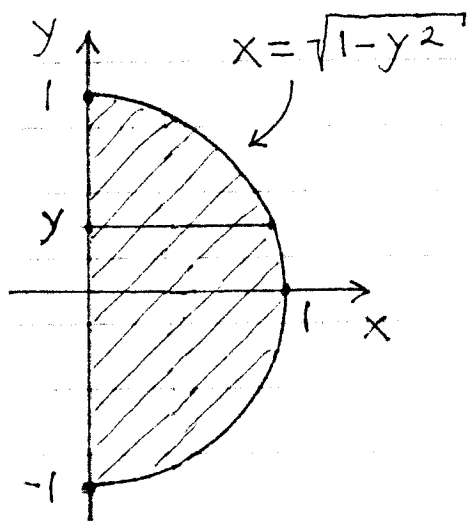
$$\text{Vol} = \int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r \, d\theta \, dz \, dr$$



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos \theta} \int_0^{3r^2} f(r, \theta, z) \cdot r \, dz \, dr \, d\theta$$

$$14.) \int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2+y^2) dz dx dy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_0^{r \cos \theta} r^2 \cdot r dz dr d\theta$$



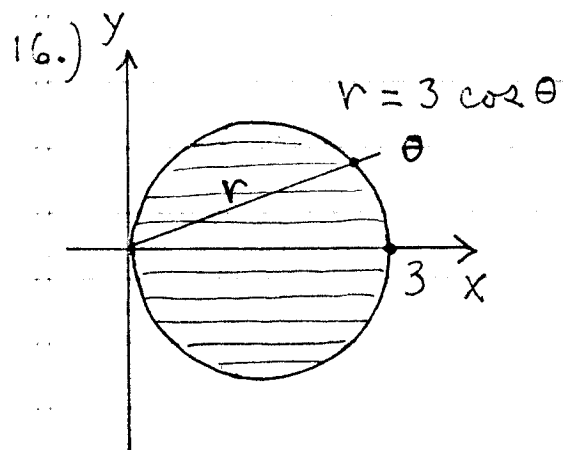
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_0^{r \cos \theta} r^3 dz dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \left(r^3 \cdot z \Big|_{z=0}^{z=r \cos \theta} \right) dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^4 \cos \theta dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{5} r^5 \cos \theta \Big|_{r=0}^{r=1} \right) d\theta$$

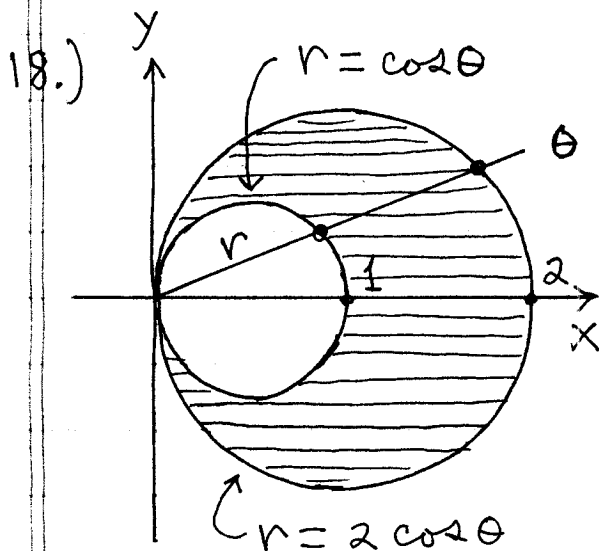
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5} \cos \theta d\theta = \frac{1}{5} \sin \theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{5} \sin \frac{\pi}{2} - \frac{1}{5} \sin \left(-\frac{\pi}{2} \right)$$

$$= \frac{1}{5}(1) - \frac{1}{5}(-1) = \frac{2}{5}$$

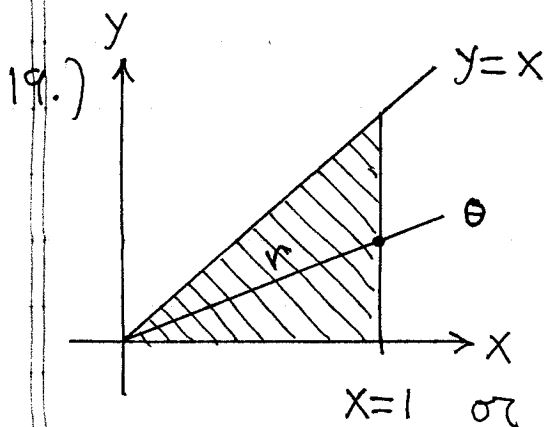


$$r = 3 \cos \theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{3 \cos \theta} \int_0^{5-r \cos \theta} f(r, \theta, z) \cdot r \, dz \, dr \, d\theta$$



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\cos \theta}^{2 \cos \theta} \int_0^{3-r \sin \theta} f(r, \theta, z) \cdot r \, dz \, dr \, d\theta$$

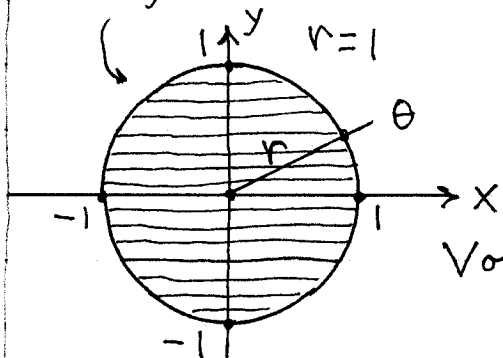


$$\int_0^{\frac{\pi}{4}} \int_0^{\sec \theta} \int_0^{2-r \sin \theta} f(r, \theta, z) \cdot r \, dz \, dr \, d\theta$$

43.)
$$\left. \begin{aligned} z &= 4 - 4(x^2 + y^2) \\ z &= (x^2 + y^2)^2 - 1 \end{aligned} \right\} \rightarrow 4(x^2 + y^2) = 4 - z \rightarrow x^2 + y^2 = 1 - \frac{1}{4}z \rightarrow$$

(SUB)
$$z = \left(1 - \frac{1}{4}z\right)^2 - 1 \rightarrow z = 1 - \frac{1}{2}z + \frac{1}{16}z^2 - 1 \rightarrow 16z = -8z + z^2 \rightarrow z^2 - 24z = 0 \rightarrow z(z - 24) = 0$$

$\rightarrow z = 0$ or $z = 24$; $0 = 4 - 4(x^2 + y^2) \rightarrow x^2 + y^2 = 1$ or $z = 4 - 4(x^2 + y^2)$ or $z = 4 - 4r^2$, $z = (x^2 + y^2)^2 - 1$ or $z = r^4 - 1$.

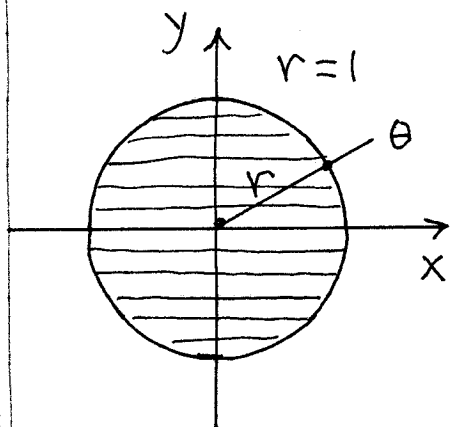


$$\text{Vol} = \int_0^{2\pi} \int_0^1 \int_{r^4-1}^{4-4r^2} r \, dz \, dr \, d\theta$$

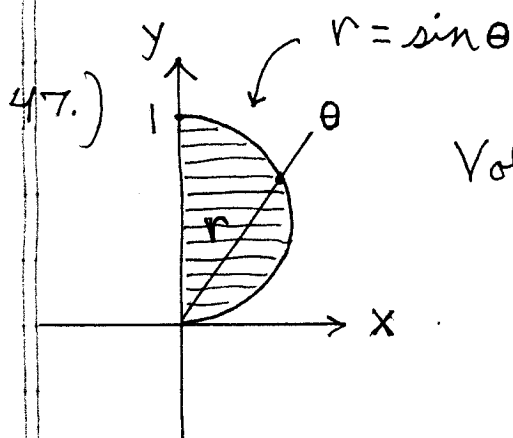
44.)
$$\left. \begin{aligned} z &= 1-r \\ z &= -\sqrt{1-r^2} \end{aligned} \right\} \rightarrow 1-r = -\sqrt{1-r^2} \rightarrow (1-r)^2 = 1-r^2 \rightarrow$$

$$1-2r+r^2 = 1-r^2 \rightarrow 2r^2-2r=0 \rightarrow$$

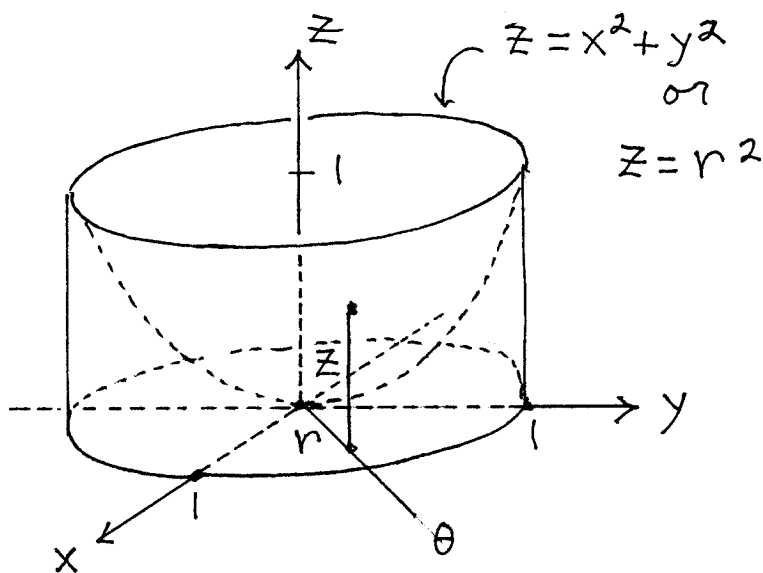
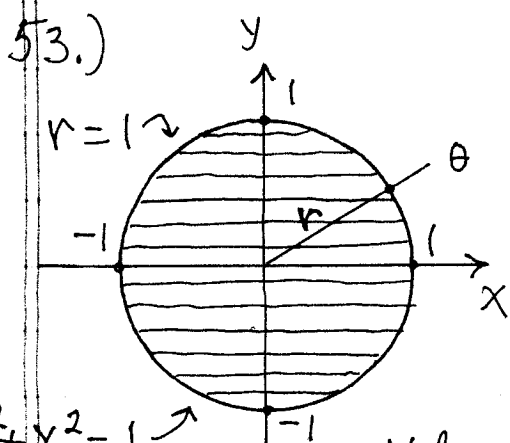
$$2r(r-1)=0 \rightarrow r \neq 0 \text{ or } r=1 \text{ and } z=0 :$$



$$Vol = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{1-r} r \, dz \, dr \, d\theta$$

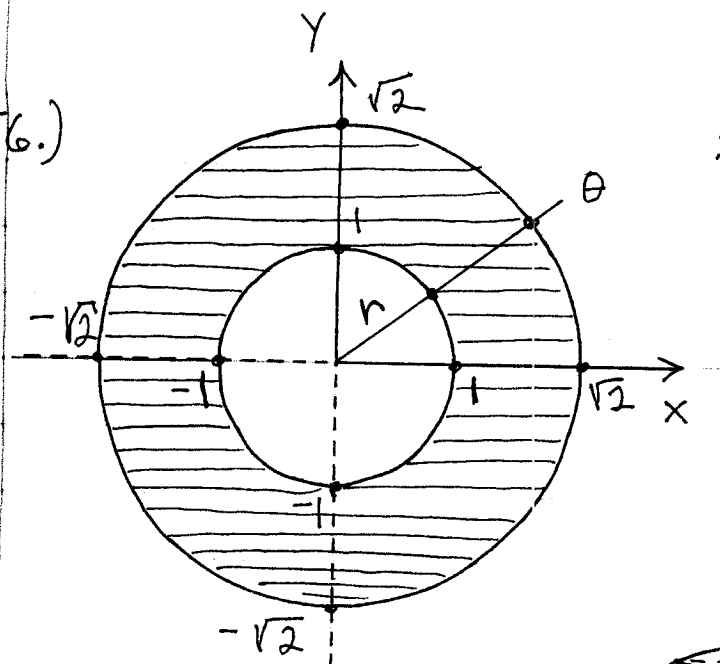


$$Vol = \int_0^{\pi/2} \int_0^{\sin \theta} \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$



$$Vol = \int_0^{2\pi} \int_0^1 \int_0^{r^2} r \, dz \, dr \, d\theta$$

56.)



$$\left. \begin{aligned} x^2 + y^2 + z^2 &= 2 \\ x^2 + y^2 &= 1 \end{aligned} \right\} \rightarrow$$

$$1 + z^2 = 2 \rightarrow z^2 = 1 \rightarrow z = \pm 1$$

$$z = \sqrt{2 - x^2 - y^2}$$

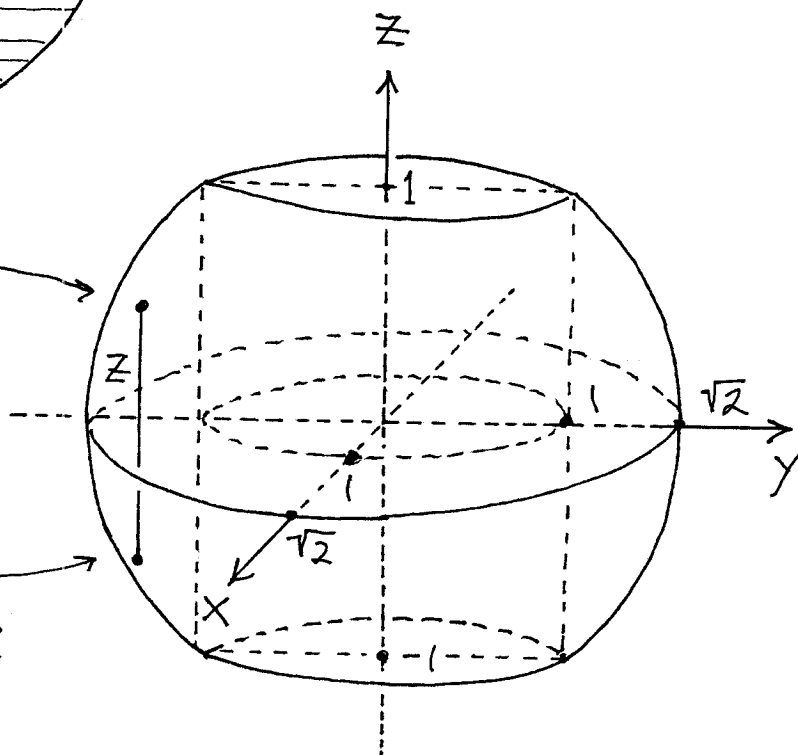
or

$$z = \sqrt{2 - r^2}$$

$$z = -\sqrt{2 - x^2 - y^2}$$

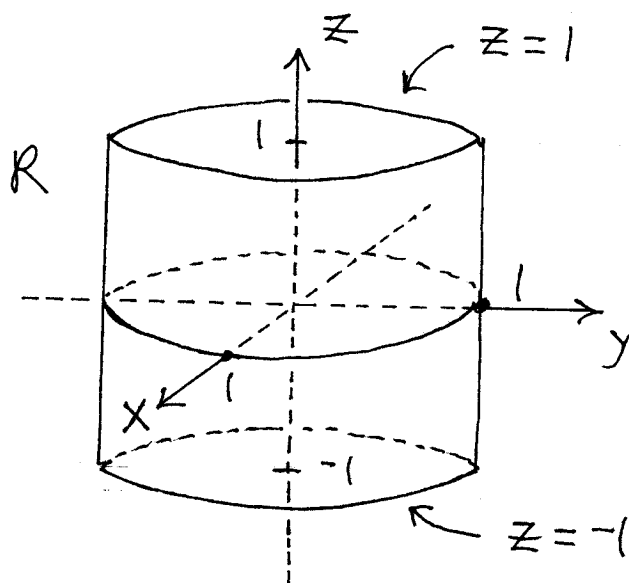
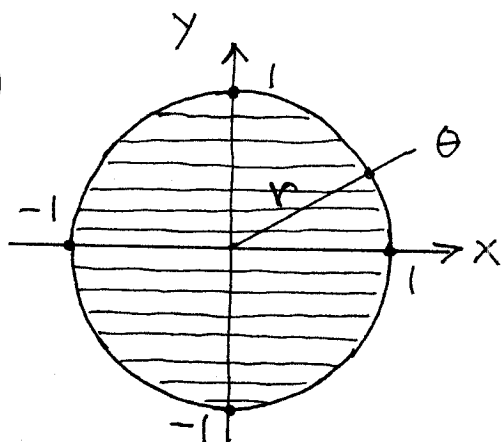
or

$$z = -\sqrt{2 - r^2}$$



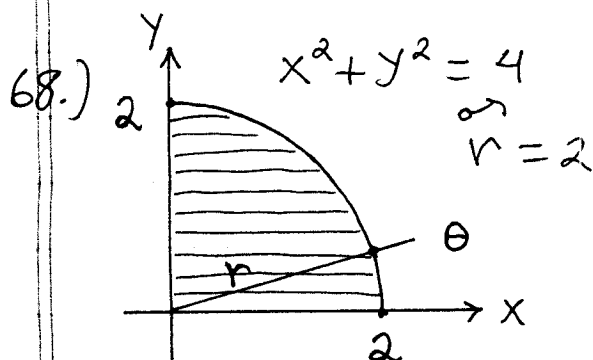
$$\text{Vol} = \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta$$

63.)



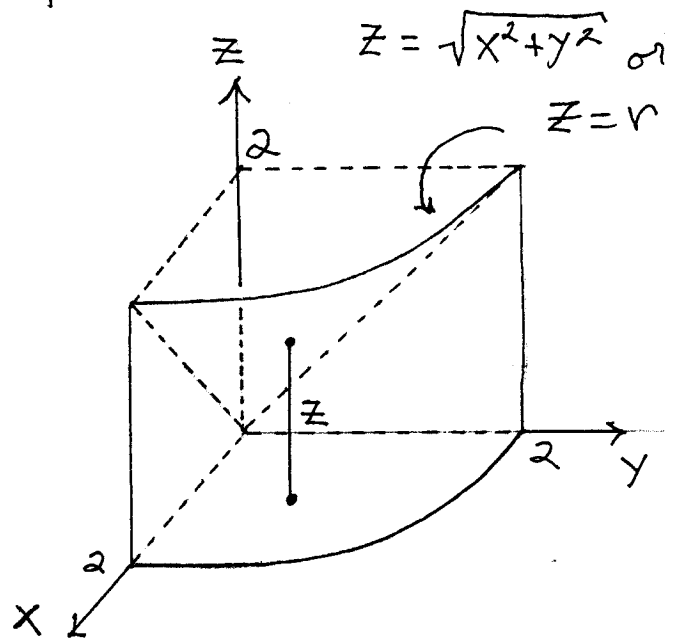
$$AVE = \frac{1}{\text{vol. } R} \cdot \iiint_R f(\rho) dV$$

$$= \frac{1}{\pi(1)^2(2)} \int_0^{2\pi} \int_0^1 \int_{-1}^1 r \cdot r dz dr d\theta$$



$$\left. \begin{array}{l} z = \sqrt{x^2 + y^2} \\ x^2 + y^2 = 4 \end{array} \right\} \rightarrow$$

$$z = \sqrt{4} = 2$$

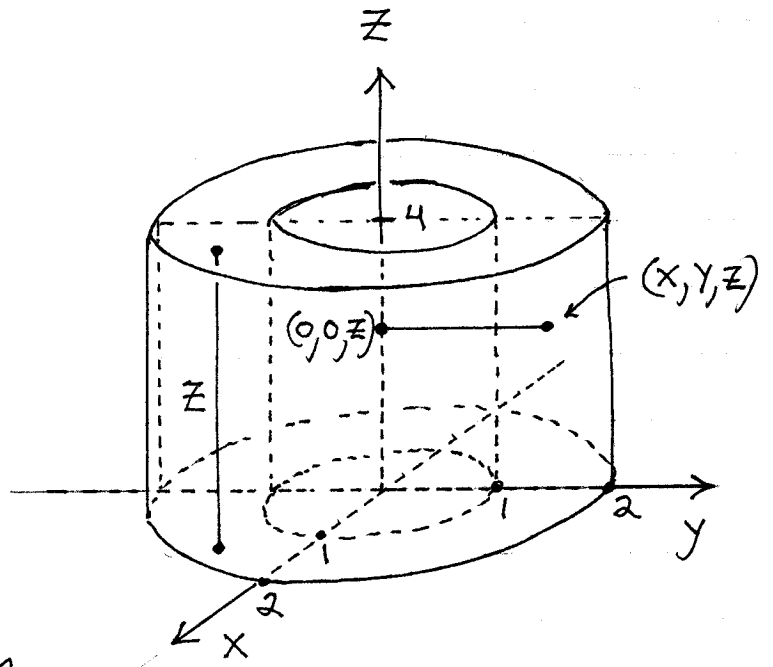
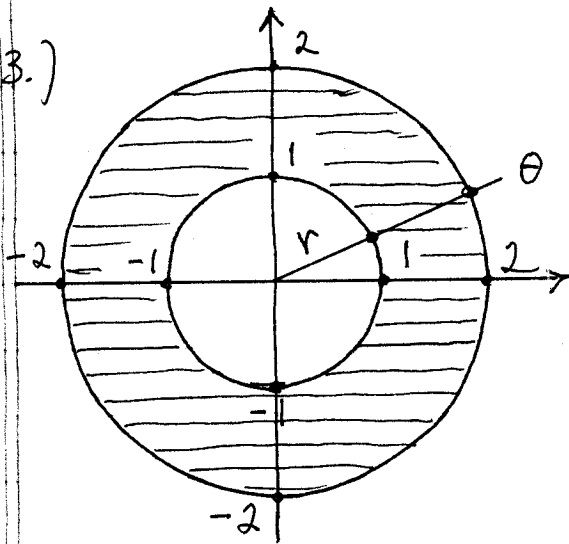


$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r (r \cos \theta) \cdot r dz dr d\theta}{\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r r dz dr d\theta},$$

$$\bar{y} = \frac{\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r (r \sin \theta) \cdot r dz dr d\theta}{\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r r dz dr d\theta},$$

$$\bar{z} = \frac{\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r z \cdot r dz dr d\theta}{\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r r dz dr d\theta}$$

73.)



distance between $(0, 0, z)$ and (x, y, z) is

$$\sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2} = \sqrt{x^2 + y^2} = \sqrt{r^2} = r ;$$

$$I_z = \iiint_R (\text{distance})^2 \delta(P) dV$$

$$= \int_0^{2\pi} \int_1^2 \int_0^4 r^2 \cdot (1) r dz dr d\theta$$