1.) (9 pts. each) Use any method to solve the following differential equations. You need not solve explicitly for y.

a.) $y' = 4x^3 + \cos 4x - \sec^2 x$

$\rightarrow \boxed{y = x^4 + \frac{1}{4}\sin 4x - \tan x + c}$

b.) $y'' = 4e^{2x} + 12x - 6$

$\rightarrow y' = 4 \cdot \frac{1}{2} e^{2x} + 6x^2 - 6x + c_1$

$\rightarrow y = 2 \cdot \frac{1}{2} e^{2x} + 6 \cdot \frac{1}{3} x^3 - 6 \cdot \frac{1}{2} x^2 + c_1 x + c_2$

$\rightarrow \boxed{y = e^{2x} + 2x^3 - 3x^2 + c_1 x + c_2}$

c.) $\frac{dy}{dx} = \sqrt{x} \cdot (y-1)^2$ $\rightarrow$ $\int \frac{1}{(y-1)^2}\, dy = \int \sqrt{x}\, dx \rightarrow$

$\boxed{-\frac{1}{y-1} = \frac{2}{3} x^{3/2} + c}$ and $\boxed{y = 1}$

d.) $y' + \frac{2}{x} \cdot y = \frac{e^x}{x}$ $\rightarrow \mu = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2 \rightarrow$

$x^2 y' + 2xy = xe^x \rightarrow D(x^2 y) = xe^x \rightarrow$

$x^2 y = \int xe^x\, dx$ (Let $u = x$, $dv = e^x dx$, $du = dx$, $v = e^x$)

$= xe^x - \int e^x\, dx = xe^x - e^x + c \rightarrow$

$\boxed{x^2 y = xe^x - e^x + c}$

1.) (12 pts.) Compute z_x, z_y, and z_{xy} for $z = x^3 + \sin y + e^{xy}$.

$$z_x = 3x^2 + ye^{xy}$$

$$z_y = \cos y + xe^{xy}$$

$$z_{xy} = \frac{\partial}{\partial y}(3x^2 + ye^{xy})$$

$$= y \cdot xe^{xy} + 1 \cdot e^{xy}$$

2.) (12 pts.) The rate at which the amount of money (\$) A in a savings account changes at time t (years) is proportional to the amount A. Initially, there is \$1000 in the account. After 5 years there is \$1500. How long will it take the account to grow to \$5000 ?

$$\frac{dA}{dt} = kA \rightarrow A = ce^{kt} \; ; \quad t=0, A = \$1000 \rightarrow$$

$$1000 = ce^0 = c \cdot 1 = c \rightarrow A = 1000e^{kt} \; ;$$

$$t = 5, A = \$1500 \rightarrow 1500 = 1000e^{5k} \rightarrow$$

$$1.5 = e^{5k} \rightarrow \ln 1.5 = \ln e^{5k} \rightarrow$$

$$\ln 1.5 = 5k \rightarrow k = \frac{1}{5}\ln 1.5 \rightarrow$$

$$\boxed{A = 1000e^{(\frac{1}{5}\ln 1.5)t}} \; ; \quad \text{if } A = \$5000 \rightarrow$$

$$5000 = 1000e^{(\frac{1}{5}\ln 1.5)t} \rightarrow 5 = e^{(\frac{1}{5}\ln 1.5)t} \rightarrow$$

$$\ln 5 = \ln e^{(\frac{1}{5}\ln 1.5)t} \rightarrow \ln 5 = (\tfrac{1}{5}\ln 1.5)t \rightarrow$$

$$t = \frac{\ln 5}{\frac{1}{5}\ln 1.5} \approx 19.8 \text{ yrs.}$$

3.) (8 pts.) Show that the equation $x^2 + xy = C$ solves the differential equation $x^2y'' - 2(x + y) = 0$.

$$x^2 + xy = C \xrightarrow{D} 2x + xy' + 1 \cdot y = 0 \rightarrow$$

$$\boxed{y' = \frac{-2x - y}{x}} \; ; \quad \xrightarrow{D} 2 + xy'' + y' + y' = 0 \rightarrow$$

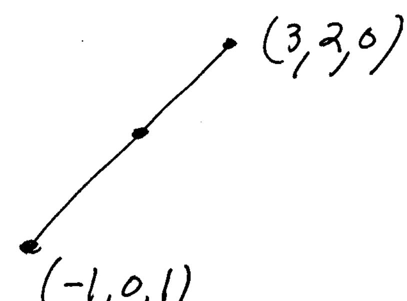
$$2 + xy'' + 2y' = 0 \rightarrow 2 + xy'' + 2\left(\frac{-2x - y}{x}\right) = 0 \rightarrow$$

$$2x + x^2 y'' - 4x - 2y = 0 \rightarrow$$

$$x^2 y'' - 2x - 2y = 0 \rightarrow$$

$$\boxed{x^2 y'' - 2(x + y) = 0} \, .$$

4.) (8 pts.) Determine an equation for the sphere which has a diameter with endpoints $(-1, 0, 1)$ and $(3, 2, 0)$.



$$\text{Dist.} = \sqrt{(3 - -1)^2 + (2-0)^2 + (0-1)^2}$$

$$= \sqrt{16 + 4 + 1} = \sqrt{21}$$

so radius is $\frac{1}{2}\sqrt{21}$;

midpoint is $\left(\frac{-1+3}{2}, \frac{0+2}{2}, \frac{1+0}{2}\right)$

$= (1, 1, \frac{1}{2})$ so center is $(1, 1, \frac{1}{2}) \rightarrow$

sphere : $(x-1)^2 + (y-1)^2 + (x - \frac{1}{2})^2 = \left(\frac{\sqrt{21}}{2}\right)^2$

5.) Consider the equation $z = 9 - x^2 - y^2$.

a.) (3 pts.) Determine the x-, y-, and z-intercepts for this surface.

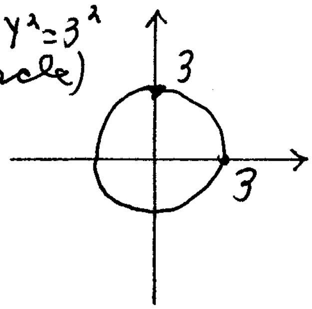
$x=0,\ y=0 \rightarrow z=9$

$x=0,\ z=0 \rightarrow y^2=9 \rightarrow y=\pm 3$

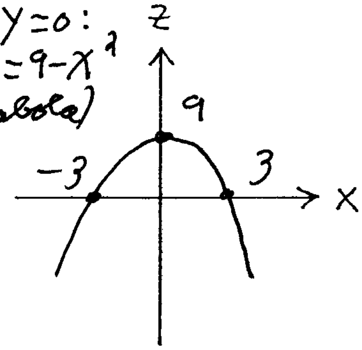
$y=0,\ z=0 \rightarrow x^2=9 \rightarrow x=\pm 3$

b.) (3 pts.) Determine and sketch the xy-, xz-, and yz-traces for this surface.

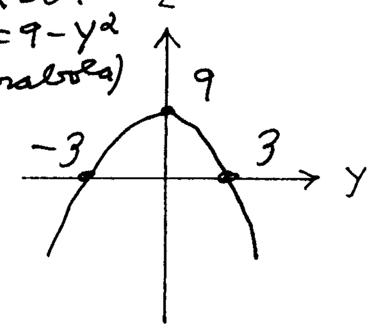
$z=0$: $x^2+y^2=3^2$ (circle)



$y=0$: $z=9-x^2$ (parabola)



$x=0$: $z=9-y^2$ (parabola)



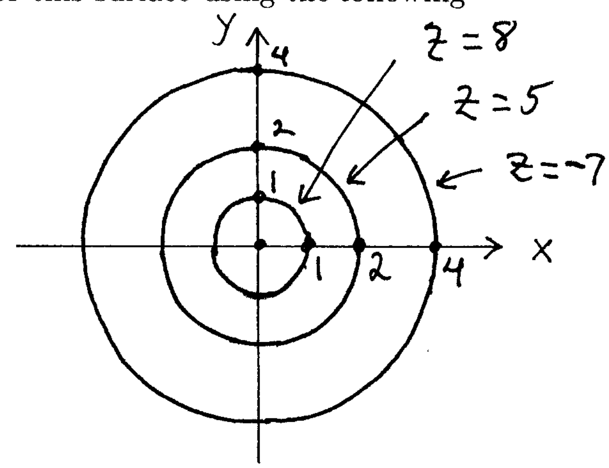
c.) (3 pts.) Determine and sketch the level curves for this surface using the following values for z : 8, 5, -7.

$z=8$: $8=9-x^2-y^2 \rightarrow x^2+y^2=1$

$z=5$: $5=9-x^2-y^2 \rightarrow x^2+y^2=2^2$

$z=-7$: $-7=9-x^2-y^2 \rightarrow x^2+y^2=4^2$

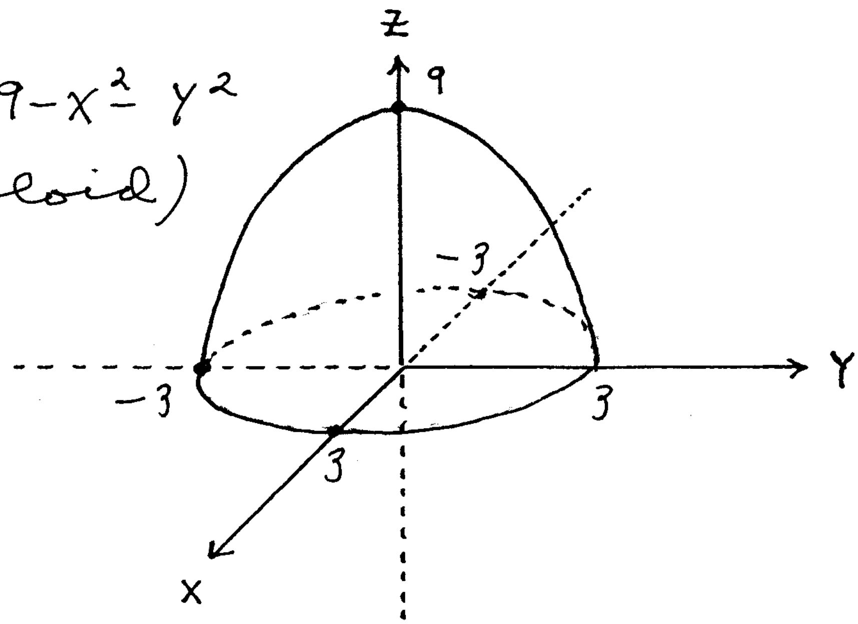
(all are circles)



d.) (3 pts.) Sketch this surface in 3D-Space.

$z=9-x^2-y^2$

(paraboloid)



6.) Water containing 2 pounds of salt per gallon flows into a tank at the rate of 4 gal./min. and the well-stirred mixture flows out of the tank at the rate of 3 gal./min. The tank initially holds 50 gallons of water containing 10 lbs. of salt. Let S represent the number of pounds of salt in the tank at time t.

a.) (5 pts.) Set up a differential equation which describes the rate at which the amount of salt changes at time t.

$$\frac{dS}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= \left(\frac{2 \text{ lbs.}}{\text{gal}}\right)\left(\frac{4 \text{ gal.}}{\text{min.}}\right) - \left(\frac{S \text{ lbs.}}{50+t \text{ gal.}}\right)\left(\frac{3 \text{ gal.}}{\text{min.}}\right) \rightarrow$$

$$\boxed{\frac{dS}{dt} = 8 - \frac{3}{50+t}\cdot S}$$

b.) (10 pts.) Solve the differential equation in part a.). Determine all unknown constants and solve explicitly for S.

$$\frac{dS}{dt} + \frac{3}{50+t}\cdot S = 8 \rightarrow \mu = e^{\int \frac{3}{50+t}\,dt} = e^{3\ln(50+t)}$$

$$= e^{\ln(50+t)^3} = (50+t)^3 \rightarrow$$

$$(50+t)^3 \cdot \frac{dS}{dt} + 3(50+t)^2 \cdot S = 8(50+t)^3 \rightarrow$$

$$D\left((50+t)^3 \cdot S\right) = 8(50+t)^3 \rightarrow$$

$$(50+t)^3 S = \int 8(50+t)^3\,dt = 2(50+t)^4 + C \rightarrow$$

$$S = 2(50+t) + \frac{C}{(50+t)^3} \;;\; \text{if } t=0,\ S=10 \text{ lbs.} \rightarrow$$

$$10 = 2(50) + \frac{C}{(50)^3} \rightarrow C = -90(50)^3 \rightarrow$$

$$\boxed{S = 100 + 2t - 90(50)^3 \cdot (50+t)^{-3}}$$