1.) Show that the function $y = c_1 x + c_2 x^3$ solves the differential equation $x^2 y'' - 3xy' + 3y = 0$ .

$$Y = c_1 X + c_2 X^3 \xrightarrow{D} y' = c_1 + 3c_2 X^2 \xrightarrow{D}$$

$$y'' = 6c_2 X \;;\; \text{then}$$

$$X^2 y'' - 3Xy' + 3Y = X^2(6c_2 X) - 3X(c_1 + 3c_2 X^2) + 3(c_1 X + c_2 X^3)$$

$$= 6c_2 X^3 - \cancel{3c_1 X} - 9c_2 X^3 + \cancel{3c_1 X} + 3c_2 X^3$$

$$= 0$$

2.) Use any method to solve the following differential equations.

a.) $y' - y^2 x e^x = 0$ (separable)

$$y' = y^2 x e^X \longrightarrow \int \frac{1}{y^2}\, dy = \int x e^x\, dx$$

(Let $u = x$, $dV = e^x dx$, $du = dx$, $v = e^x$)

$$\rightarrow \frac{-1}{y} = xe^X - \int e^x\, dx \rightarrow \frac{-1}{y} = xe^X - e^X + C$$

b.) $\dfrac{dy}{dx} + \dfrac{2y}{x} = 3x + 1$ (1st-order linear)

$$y' + \left(\frac{2}{X}\right) y = 3X + 1 \rightarrow \mu = e^{\int \frac{2}{X} dx} = e^{2\ln X} = e^{\ln X^2} = X^2 \rightarrow$$

$$X^2 y' + 2Xy = 3X^3 + X^2 \longrightarrow D(X^2 y) = 3X^3 + X^2 \longrightarrow$$

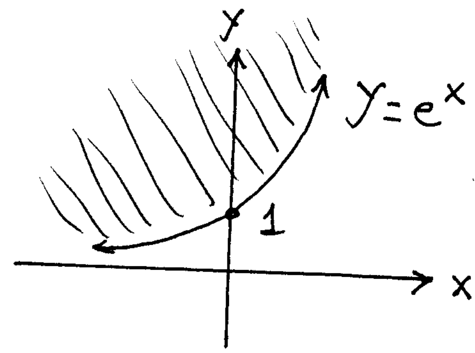
$$X^2 y = \int (3X^3 + X^2)\, dx$$

$$= \frac{3}{4} X^4 + \frac{X^3}{3} + C \rightarrow X^2 y = \frac{3}{4} X^4 + \frac{X^3}{3} + C$$

3.) Consider the function $f(x,y) = 3\sin\sqrt{y - e^x}$ .

a.) Determine the domain of $f$. Sketch the domain in two-dimensional space.

$y - e^x \geq 0 \rightarrow$ domain is all pts. $(x, y)$ satisfying $y \geq e^x$.




b.) Determine the range of $f$. Briefly explain.

Since $-1 \leq \sin w \leq +1$, it follows that range for $f(x,y) = 3\sin\sqrt{y-e^x}$ is

Range : $-3 \leq z \leq +3$

4.) Use Lagrange Multipliers to minimize the function $w = x^2 + y^2 + z^2$ subject to the constraint $x + 2y - z = 6$ .

Let $F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + 2y - z - 6) \rightarrow$

$$\left.\begin{aligned} F_x &= 2x - \lambda = 0 \rightarrow \lambda = 2x \\ F_y &= 2y - 2\lambda = 0 \rightarrow \lambda = y \\ F_z &= 2z + \lambda = 0 \rightarrow \lambda = -2z \end{aligned}\right\} \quad \underline{y = 2x}, \text{ and } -2z = 2x \rightarrow \underline{z = -x}$$

$F_\lambda = -x - 2y + z + 6 = 0 \longrightarrow$ (SUB)

$-x - 2(2x) + (-x) + 6 = 0 \rightarrow$

$-x - 4x - x + 6 = 0 \rightarrow$

$-6x + 6 = 0 \rightarrow$

$x = 1$, $y = 2$, $z = -1$, $(\lambda = 2)$, and

min. $w = (1)^2 + (-2)^2 + (-1)^2 = 6$.

5.) Find and classify the critical points as determining relative maximum, relative minimum, or saddle points.

$$z = 2y^3 - 3x^2 - 3xy + 9x$$

$z_x = -6x - 3y + 9 = 0 \rightarrow 3(-2x - y + 3) = 0 \rightarrow$

$-2x - y + 9 = 0 \rightarrow \boxed{y = 3 - 2x}$ ;

$z_y = 6y^2 - 3x = 0 \rightarrow 3(2y^2 - x) = 0 \rightarrow \boxed{x = 2y^2}$ ;

(SUB) $y = 3 - 2(2y^2) \rightarrow y = 3 - 4y^2 \rightarrow$

$4y^2 + y - 3 = 0 \rightarrow (y+1)(4y-3) = 0 \rightarrow$

$y = -1$ or $y = 3/4$ so critical points are

$(2,-1)$ and $(9/8, 3/4)$ ;

$z_{xx} = -6$, $z_{yy} = 12y$, $z_{xy} = -3$ then

For $(2,-1)$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (-6)(-12) - (-3)^2 = 63 > 0$ and $z_{xx} = -6 < 0$ so $(2,-1)$ determines a maximum value of $z = 10$.

For $(9/8, 3/4)$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (-6)(9) - (-3)^2 = -63 < 0$ so $(9/8, 3/4)$ determines a saddle point at $z = \frac{297}{64}$.

6.) Find the interval of convergence and radius of convergence for the following power series :

$$\sum_{n=1}^{\infty} \frac{2^{n+1}(x-1)^n}{n^3} \; ; \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+2}|x-1|^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^{n+1}|x-1|^n}$$

$$= \lim_{n\to\infty} 2 \cdot \left(\frac{n}{n+1}\right)^3 \cdot |x-1| = \lim_{n\to\infty} 2\left(\frac{1}{1+\frac{1}{n}}\right)^3 \cdot |x-1|$$

$$= 2\left(\frac{1}{1+0}\right)^3 |x-1| = 2(1)|x-1| = 2|x-1| < 1 \rightarrow$$

$$|x-1| < \frac{1}{2} \rightarrow \frac{-1}{2} < x - 1 < \frac{1}{2} \rightarrow \frac{1}{2} < x < \frac{3}{2}$$

and $R = \frac{1}{2}$.

7.) Find the 300th number in the following sequence :
6, 9, 13, 18, 24, 31, $\cdots$

| $n$ | # |
|---|---|
| 1 | $6 = (1+2)+3$ |
| 2 | $9 = (1+2+3)+3$ |
| 3 | $13 = (1+2+3+4)+3$ |
| 4 | $18 = (1+2+3+4+5)+3$ |
| $\vdots$ | $\vdots$ |
| $n$ | $(1+2+3+\cdots+(n+1))+3$ |

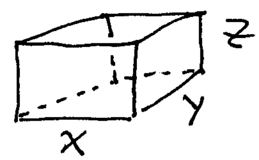
$$= \frac{(n+1)((n+1)+1)}{2} + 3$$
$$= \frac{(n+1)(n+2)}{2} + 3\ ;$$

if $n = 300 \rightarrow$ # is

$$\frac{(301)(302)}{2} = 45{,}451$$

8.) Build an open (no top) rectangular box with a volume of 32 ft.$^3$ What dimensions will minimize the surface area of the box ? (Find the critical point, but you need not verify that it determines a minimum value.)

Volume $xyz = 32 \rightarrow z = \dfrac{32}{xy}$ ;




minimize surface area

$$S = xy + 2xz + 2yz = xy + 2x\cdot\left(\frac{32}{xy}\right) + 2y\cdot\left(\frac{32}{xy}\right) \rightarrow$$

$S = xy + \dfrac{64}{y} + \dfrac{64}{x}$ ; then

$$\left.\begin{array}{l} S_x = y - \dfrac{64}{x^2} = 0 \rightarrow y = \dfrac{64}{x^2} \\ S_y = x - \dfrac{64}{y^2} = 0 \rightarrow x = \dfrac{64}{y^2} \end{array}\right\} \rightarrow x = \frac{64}{\left(\frac{64}{x^2}\right)^2} = \frac{1}{64}x^4 \rightarrow$$

$64x = x^4 \rightarrow x^4 - 64x = 0 \rightarrow$
$x(x^3 - 64) = 0 \rightarrow x = 0$ (no)

or $x = 4$ ft., $y = 4$ ft., $z = 2$ ft., $S = 48$ ft.$^2$

9.) What should $n$ be in order that the Taylor Polynomial $P_n(x)$ of degree $n$ centered at $c = 0$ estimate the value of $\dfrac{1}{1-x}$ with absolute error at most 0.0001 at the point $x = 1/3$.

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$
$$f'(x) = -(1-x)^{-2}(-1) = (1-x)^{-2}$$
$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$
$$f'''(x) = -3\cdot 2\,(1-x)^{-4}(-1) = 3\cdot 2(1-x)^{-4}$$
$$f^{(4)}(x) = 4\cdot 3\cdot 2\,(1-x)^{-5}\ \ldots,$$
$$f^{(n)}(x) = n!\,(1-x)^{-(n+1)},$$
$$f^{(n+1)}(x) = (n+1)!\,(1-x)^{-(n+2)}$$

$$|f(x) - P_n(x)| \le \left|\frac{f^{(n+1)}(z)\,(x-c)^{n+1}}{(n+1)!}\right|$$

(where $z$ is between $x = \frac{1}{3}$ and $c = 0$)

$$\le \frac{\cancel{(n+1)!}\,|1-z|^{-(n+2)}\,\left|\frac{1}{3}-0\right|^{n+1}}{\cancel{(n+1)!}}$$
$$= \frac{\left(\frac{1}{3}\right)^{n+1}}{|1-z|^{n+2}} \le \frac{\left(\frac{1}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^{n+1}}$$
$$= \frac{3}{2}\left(\frac{1}{2}\right)^{n+1} \le 0.0001 \rightarrow$$
$$\left(\frac{1}{2}\right)^{n+1} \le \frac{2}{3}(0.0001) \rightarrow$$

$(n+1)\ln\left(\frac{1}{2}\right) \le \ln\left(\frac{2}{3}(0.0001)\right) \rightarrow (n+1) > \dfrac{\ln 2/3\,(0.0001)}{\ldots} \rightarrow n \ge 12.87 \rightarrow$

10.) Use any method to find the first three (3) nonzero terms of the Taylor Series centered at the given value of $c$ for each function.

a.) $f(x) = e^{-x} \cdot \cos\sqrt{x}$, $c = 0$

$$e^{-x} \cdot \cos\sqrt{x} = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{\sqrt{x}^2}{2!} + \frac{\sqrt{x}^4}{4!} - \frac{\sqrt{x}^6}{6!} + \cdots\right)$$

$$= \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots\right)\left(1 - \frac{x}{2} + \frac{x^2}{24} - \cdots\right)$$

$$= 1 - \frac{x}{2} + \frac{x^2}{24} - \cdots$$
$$- x + \frac{x^2}{2} - \cdots$$
$$+ \frac{x^2}{2} - \cdots$$

$$= 1 - \frac{3}{2}x + \frac{25}{24}x^2 - \cdots$$

b.) $f(x) = \sqrt{x}$, $c = 4$

$$f'(x) = \frac{1}{2}x^{-1/2}$$
$$f''(x) = -\frac{1}{4}x^{-3/2}$$

$$a_n = \frac{f^{(n)}(c)}{n!} = \frac{f^{(n)}(4)}{n!}$$

then $a_0 = \dfrac{f(4)}{0!} = \dfrac{\sqrt{4}}{1} = 2$,

$$a_1 = \frac{f'(4)}{1!} = \frac{\frac{1}{2}(4)^{-1/2}}{1} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{1} = \frac{1}{4},$$

$$a_2 = \frac{f''(4)}{2!} = \frac{-\frac{1}{4}(4)^{-3/2}}{2} = \frac{-\frac{1}{4}\cdot\frac{1}{8}}{2} = \frac{-1}{64};$$ so

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \cdots$$

11.) Use the 6th-degree Taylor Polynomial $P_6(x)$ centered at $c = 0$ for $\frac{x^5}{1-x}$ to estimate the value of $\int_0^{1/2} \frac{x^5}{1-x}\, dx$

$$\frac{x^5}{1-x} = x^5 \cdot \frac{1}{1-x} = x^5(1 + x + x^2 + \cdots) = x^5 + x^6 + x^7 + \cdots$$

so $P_6(x) = x^5 + x^6$ and

$$\int_0^{1/2} \frac{x^5}{1-x}\,dx \approx \int_0^{1/2} (x^5 + x^6)\,dx = \left(\frac{x^6}{6} + \frac{x^7}{7}\right)\Big|_0^{1/2}$$

$$= \frac{1}{6}\left(\frac{1}{2}\right)^6 + \frac{1}{7}\left(\frac{1}{2}\right)^7 = \frac{1}{384} + \frac{1}{896} \approx 0.0037$$

12.) Evaluate the following double intergrals.

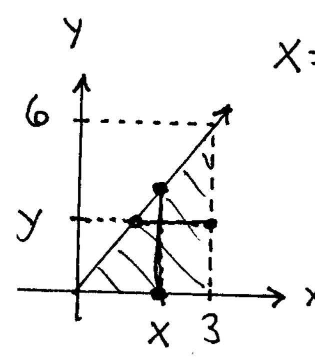
a.) $\int_0^{\sqrt{\pi/2}} \int_0^{y^2} 2y\cos(x)\; dx dy$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} \left(2y \sin x \Big|_{x=0}^{x=y^2}\right) dy = \int_0^{\sqrt{\frac{\pi}{2}}} \left(2y \sin y^2 - 2y \cancel{\sin 0}^{\,0}\right) dy$$

$$= -\cos y^2 \Big|_0^{\sqrt{\frac{\pi}{2}}} = -\cos\frac{\pi}{2} - -\cos 0$$

$$= -(0) + 1 = 1$$

b.) $\int_0^6 \int_{y/2}^3 e^{x^2}\, dx dy$ (SWITCH ORDER)




$X = Y/2$ or $Y = 2X$

$$= \int_0^3 \int_0^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^3 \left(y e^{x^2} \Big|_{y=0}^{y=2x}\right) dx$$

$$= \int_0^3 2x e^{x^2}\, dx$$

$$= e^{x^2}\Big|_0^3 = e^9 - e^0 = e^9 - 1$$

13.) Use Netwon's Method to estimate the solution of $x^4 - 3x = 8$. Start with $x_1 = 1$ and compute the next two Newton approximations, $x_2$ and $x_3$.

$x^4 - 3x = 8 \rightarrow x^4 - 3x - 8 = 0$ so let

$f(x) = x^4 - 3x - 8 \xrightarrow{D} f'(x) = 4x^3 - 3$ so

Newton : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 3x_n - 8}{4x_n^3 - 3}$

$= \frac{4x_n^4 - \cancel{3x_n} - x_n^4 + \cancel{3x_n} + 8}{4x_n^3 - 3} \rightarrow \boxed{x_{n+1} = \frac{3x_n^4 + 8}{4x_n^3 - 3}}$ ;

$x_1 = 1 \rightarrow x_2 = \frac{3x_1^4 + 8}{4x_1^3 - 3} = \frac{3(1)^4 + 8}{4(1)^3 - 3} = \frac{11}{1} = 11$ ;

$x_3 = \frac{3x_2^4 + 8}{4x_2^3 - 3} = \frac{3(11)^4 + 8}{4(11)^3 - 3} \approx 8.256$

14.) A tank is holding 50 gallons of pure water. A solution containg 1/2 pound of salt per gallon begins flowing into the tank at the rate of 3 gallons per minute. At the same time the well-stirred mixture begins flowing out of the tank at the rate of 4 gallons per minute. How much salt is in the tank after $t = 10$ minutes ?

Let $S$ : lbs. of salt in tank at time $t$ (min.);

$\frac{dS}{dt} = (\text{rate in}) - (\text{rate out}) = \left(\frac{1}{2}\frac{\text{lb.}}{\text{gal.}}\right)\left(3\frac{\text{gal.}}{\text{min.}}\right) - \left(\frac{S\text{ lbs.}}{50-t}\right)\left(\frac{4\text{ gal.}}{\text{min.}}\right)$

$\rightarrow \frac{dS}{dt} = \frac{3}{2} - \frac{4}{50-t}\cdot S \rightarrow \boxed{\frac{dS}{dt} + \frac{4}{50-t}\cdot S = 3/2}$ ; then

$\mu = e^{\int \frac{4}{50-t}\,dt} = e^{-4\ln(50-t)} = e^{\ln(50-t)^{-4}} = (50-t)^{-4} \rightarrow$

$(50-t)^{-4}\cdot S' + 4(50-t)^{-5}\cdot S = \frac{3}{2}(50-t)^{-4} \rightarrow$

$D\{(50-t)^{-4}\cdot S\} = \frac{3}{2}(50-t)^{-4} \rightarrow (50-t)^{-4}\cdot S = \int \frac{3}{2}(50-t)^{-4}\,dt$

$\rightarrow (50-t)^{-4}\cdot S = \frac{\cancel{3}}{2}\cdot\frac{1}{\cancel{3}}(50-t)^{-3} + C \rightarrow \underline{S = \frac{1}{2}(50-t) + C(50-t)^4}$ and

$t = 0,\ S = 0$ lbs. $\rightarrow 0 = 25 + C\cdot 50^4 \rightarrow C = -0.000004 \rightarrow$

$\boxed{S = \frac{1}{2}(50-t) - 0.000004(50-t)^4}$ ; if $t = 10$ min.

then $S = 9.76$ lbs.

The following EXTRA CREDIT PROBLEM is worth This problem is OPTIONAL.

1.) Determine the maximum amount of salt in the tank in problem 14.

Maximize $S = \frac{1}{2}(50-t) - 0.000004(50-t)^4$

$\xrightarrow{D} S' = -\frac{1}{2} - 0.000004(4)(50-t)^3(-1)$

$= -\frac{1}{2} + 0.000016(50-t)^3 = 0 \rightarrow$

$\frac{1}{2} = 0.000016(50-t)^3 \rightarrow$

$31{,}250 = (50-t)^3 \rightarrow$

$31{,}250^{1/3} = 50 - t \rightarrow$

$t = 50 - 31{,}250^{1/3} \rightarrow$

$t \approx 18.5$ min.

$0$

$t = 18.5$ min. $\quad S'$

max. $S \approx 11.81$ lbs.