EXAMPLE: Assume that y, the height (ft.) of a tree, is a function of x, its base diameter (ft.). Assume also that experimental data has determined that the rate at which y changes with respect to x is proportional to its height y. Data shows that if x=1 ft., then y=10 ft. and if x=2 ft., then y=15 ft. What is the expected height y of a tree having a base diameter of x=4 ft. ?

1.) Write an equation for the Differential Equation.
2.) Solve the Differential Equation.
3.) Write y explicitly as a function of x.

Example : Let S represent the amount (pounds) of salt in a tank at time t (minutes). A solution containing 2 lbs. of salt per gallon flows into the tank at the rate of 3 gal./min., and the well-stirred mixture flows out at the rate of 5 gal./min. Assume that the tank initially holds 500 gallons of solution containing 25 lbs. of salt.

Q1 : In how many minutes will the tank be empty ?

Q2 : How much salt is in the tank after 10 minutes ? 100 minutes ? 200 minutes ? 240 minutes ?

Q3 : What is the maximum amount of salt in the tank ?

Example : Assume that the number A of insects in a small population at time t (weeks) changes at a rate proportional to the square of the number present. If initially there are 250 insects and after 3 weeks there are 420, how many insects do you expect after 7 weeks ?

$$\frac{dA}{dt} = kA^2 \rightarrow \int \frac{1}{A^2}\, dA = \int k\, dt \rightarrow$$

$$\frac{-1}{A} = kt + c \rightarrow A = \frac{-1}{kt+c} \;;$$

if $t=0$, $A=250$ then $250 = \frac{-1}{c} \rightarrow c = \frac{-1}{250} \rightarrow$

$$A = \frac{-1}{kt - \frac{1}{250}} = \frac{250}{-250kt+1} \;;$$

if $t=3$, $A=420$ then $420 = \frac{250}{-750k+1} \rightarrow$

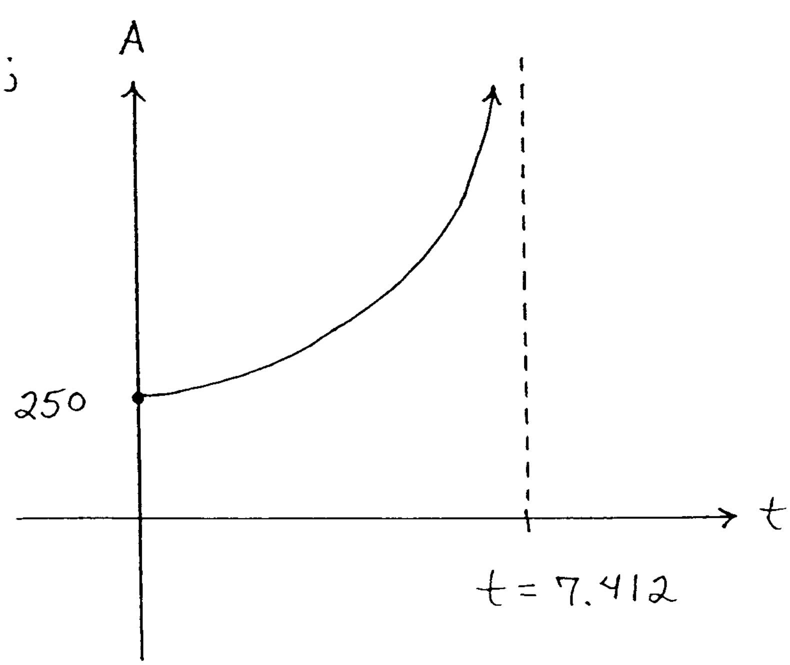
$$k = \frac{1}{750}\left(1 - \frac{250}{420}\right) = \frac{17}{31{,}500} = 0.000539682 \rightarrow$$

$$A = \frac{250}{1-0.134920634\,t} \;;$$

if $t = 7$ weeks

then $A \approx 4500$

insects .



Example : Assume that the number P of elk at time t (years) on a large game preserve changes at a rate proportional to the product of P and 500 − P . Initially there are 85 elk and after 5 years there are 130. How many elk do you expect after 10 years ?

$$\frac{dA}{dt} = kP(500-P) \rightarrow \int \frac{1}{P(500-P)}\,dP = \int k\,dt \rightarrow$$

$$\int \left[\frac{\frac{1}{500}}{P} + \frac{\frac{1}{500}}{500-P} \right] dP = kt + c \rightarrow$$

$$\frac{1}{500}\ln P - \frac{1}{500}\ln(500-P) = kt + c \rightarrow$$

$$\ln P - \ln(500-P) = 500kt + 500c \rightarrow$$

$$\ln \frac{P}{500-P} = 500kt + c \rightarrow$$

$$\frac{P}{500-P} = e^{c+500kt} = e^{c}e^{500kt} = ce^{500kt} \rightarrow$$

$$\frac{P}{500-P} = ce^{500kt} \text{ ; if } t=0,\ P=85 \text{ then}$$

$$\frac{85}{415} = c = 0.204819277 \rightarrow \frac{P}{500-P} = 0.204819277 \cdot e^{500kt} \text{ ;}$$

$$\text{if } t=5,\ P=130 \text{ then } \frac{130}{370} = 0.204819277 \cdot e^{2500k} \rightarrow$$

$$k = 0.000215863 \quad (\text{why ?}) \rightarrow$$

$$\frac{P}{500-P} = 0.204819277 \cdot e^{0.1079315t} \text{ ;}$$

Solve explicitly for P. First,

$$P = 102.4096385\, e^{0.1079315t} - 0.204819277 \cdot e^{0.1079315t} \cdot P \rightarrow$$

$$\left(1 + 0.204819277 \cdot e^{0.1079315t}\right) P = 102.4096385 \cdot e^{0.1079315t} \rightarrow$$

$$P = \frac{102.4096385 \cdot e^{0.1079315t}}{1 + 0.204819277 \cdot e^{0.1079315t}} \rightarrow$$

$$P = \frac{102.4096385}{e^{-0.1079315t} + 0.204819277} \, ;$$

if $t = 10$ years then $P \approx 188$ elk.

Note that $\lim_{t \to +\infty} P = 500$ (why?).