Math 16C
Vogler

Geometric Series

Fact 1 $1+r+r^2+r^3+\dots+r^n = \dfrac{1-r^{n+1}}{1-r}$ for any constant $r \neq 1$

Pf Let $S = 1+r+r^2+r^3+\dots+r^n$

$$\Rightarrow \quad rS = r+r^2+r^3+\dots+r^n+r^{n+1}$$

Subtracting $S - rS = 1 - r^{n+1} \Rightarrow (1-r)S = 1-r^{n+1} \Rightarrow S = \dfrac{1-r^{n+1}}{1-r}$ ∎

Fact 2

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \\ \text{DNE} & \text{if } r\leq 1 \end{cases}$$

Fact 3 $1+r+r^2+r^3+\dots = \dfrac{1}{1-r}$ for $-1<r<1$.
The infinite series diverges for all other values of r.

Pf $1+r+r^2+r^3+\dots = \lim\limits_{n\to\infty}(1+r+r^2+\dots+r^n)$

$= \lim\limits_{n\to\infty} \dfrac{1-r^{n+1}}{1-r} = \dfrac{1-0}{1-r} = \dfrac{1}{1-r}$ for $-1<r<1$. For other values of r, the limit is infinite or does not exist ∎

These facts lead to the following result

Thm The infinte geometric series converges to

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{when } |r|<1.$$

When $|r| \geq 1$ the series $\sum\limits_{n=0}^{\infty} ar^n$ diverges.