

Taylor's Theorem

Given a function $f(x)$ & a constant c , find constants $a_0, a_1, a_2, a_3, \dots, a_n, \dots$ such that

$$(1) f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots + a_n(x-c)^n + \dots$$

$$f(c) = a_0 + a_1(0) + a_2(0) + \dots = a_0 \quad \text{so}$$

$$a_0 = f(c); \quad \text{Differentiate (1)} \Rightarrow$$

$$(2) f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \dots + na_n(x-c)^{n-1} + \dots$$

$$f'(c) = a_1 + 2a_2(0) + 3a_3(0) + \dots = a_1 \quad \text{so}$$

$$a_1 = f'(c); \quad \text{Differentiate (2)} \Rightarrow$$

$$(3) f''(x) = 2a_2 + 3 \cdot 2a_3(x-c) + 4 \cdot 3a_4(x-c)^2 + \dots + n(n-1)a_n(x-c)^{n-2} + \dots$$

$$f''(c) = 2a_2 + 3 \cdot 2a_3(0) + 4 \cdot 3a_4(0) + \dots = 2a_2 \quad \text{so}$$

$$a_2 = \frac{f''(c)}{2}; \quad \text{Differentiate (3)} \Rightarrow$$

$$(4) f'''(x) = 3 \cdot 2a_2 + 4 \cdot 3 \cdot 2a_4(x-c) + 5 \cdot 4 \cdot 3a_5(x-c)^2 + \dots$$

$$f'''(c) = 3 \cdot 2a_2 + 4 \cdot 3 \cdot 2a_4(0) + 5 \cdot 4 \cdot 3a_5(0) + \dots = 3 \cdot 2a_2 \quad \text{so}$$

$$a_3 = \frac{f'''(c)}{3 \cdot 2}; \quad \text{Differentiate (4)} \Rightarrow$$

$$(5) f^{(4)}(x) = 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5(x-c) + 6 \cdot 5 \cdot 4 \cdot 3a_6(x-c)^2 + \dots$$

$$f^{(4)}(c) = 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5(0) + 6 \cdot 5 \cdot 4 \cdot 3a_6(0) + \dots = 4 \cdot 3 \cdot 2a_4$$

$$\text{so } a_4 = \frac{f^{(4)}(c)}{4 \cdot 3 \cdot 2}$$

Continuing this process, it follows that

$$a_n = \frac{f^{(n)}(c)}{n!} \quad \text{for } n = 0, 1, 2, 3, 4, \dots$$

Summarizing, we have:

Taylor's Theorem

If $f(x)$ is a function and c is a constant where

$$(*) \quad f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots + a_n(x-c)^n + \dots$$

$$\Rightarrow \quad \boxed{a_n = \frac{f^{(n)}(c)}{n!}} \quad \text{for } n = 0, 1, 2, 3, 4, \dots$$

The series in equation (*) is called the
Taylor series centered at c for the function $f(x)$.

Note: A Taylor series centered at 0 is sometimes
~~referred~~ referred to as a Maclaurin series.