Section 7.6

18.) $f(x,y,z) = x^2y^2z^2$ and $x^2+y^2+z^2-1=0$, then

$$F(x,y,z,\lambda) = x^2y^2z^2 - \lambda(x^2+y^2+z^2-1)$$
$$= x^2y^2z^2 - \lambda x^2 - \lambda y^2 - \lambda z^2 + \lambda \rightarrow$$

$F_x = 2xy^2z^2 - 2\lambda x = 2x(y^2z^2-\lambda) = 0$
$\rightarrow x=0$ (No) or $\lambda = y^2z^2$,

$F_y = 2x^2yz^2 - 2\lambda y = 2y(x^2z^2-\lambda) = 0$
$\rightarrow y=0$ (No) or $\lambda = x^2z^2$,

$F_z = 2x^2y^2z - 2\lambda z = 2z(x^2y^2-\lambda) = 0$
$\rightarrow z=0$ (No) or $\lambda = x^2y^2$;

$F_\lambda = -x^2-y^2-z^2+1=0$;

$$\begin{cases} y^2z^2 = x^2z^2 \rightarrow y^2=x^2 \rightarrow y=x \\ x^2z^2 = x^2y^2 \rightarrow z^2=y^2 \rightarrow z=y \end{cases}$$

$z=y=x$

$\rightarrow -x^2-(x)^2-(x)^2+1=0$
$\rightarrow -3x^2+1=0$
$\rightarrow 3x^2=1 \rightarrow x^2=\frac{1}{3} \rightarrow x = 1/\sqrt{3}$, $y = 1/\sqrt{3}$,
$z = 1/\sqrt{3}$, $\lambda = 1/9$, and maximum value
$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{27}.$$

22.) $f(x,y,z) = x^2+y^2+z^2$ and
$x+2z-4=0$, $x+y-8=0$, then

$$F(x,y,z,\lambda,\mu) = x^2+y^2+z^2 - \lambda(x+2z-4) - \mu(x+y-8)$$
$$= x^2+y^2+z^2 - \lambda x - 2\lambda z + 4\lambda - \mu x - \mu y + 8\mu \rightarrow$$

$F_x = 2x - \lambda - \mu = 0 \longrightarrow 2x - (z) - (2y) = 0$
$F_y = 2y - \mu = 0 \rightarrow \mu = 2y$
$F_z = 2z - 2\lambda = 0 \rightarrow \lambda = z$
$F_\lambda = -x - 2z + 4 = 0 \rightarrow$
$F_\mu = -x - y + 8 = 0 \rightarrow$

$$\begin{cases} 2x - 2y - z = 0 \\ -x \quad - 2z = -4 \\ -x - y \quad = -8 \end{cases}$$

$$\rightarrow \begin{cases} 2x-2y-z=0 \\ -2x \quad -4z=-8 \\ -2x-2y \quad =-16 \end{cases} \rightarrow \begin{cases} -2y-5z=-8 \\ -4y-z=-16 \end{cases}$$

$$\rightarrow \begin{cases} 4y+10z=16 \\ -4y-z=-16 \end{cases} \rightarrow 9z=0 \rightarrow z=0,$$

$y=4$, $x=4$, $\lambda=0$, and minimum value $f(4,4,0)=16+16+0=32$.

24.) $f(x,y,z)=xy+yz$ and

$x+2y-6=0$, $x-3z=0$; then

$$F(x,y,z,\lambda,\mu)=xy+yz-\lambda(x+2y-6)-\mu(x-3z)$$
$$=xy+yz-\lambda x-2\lambda y+6\lambda-\mu x+3\mu z \rightarrow$$

$$\begin{cases} F_x=y-\lambda-\mu=0 \rightarrow y=\lambda+\mu = \frac{1}{2}x+\frac{1}{2}z-\frac{1}{3}y \\ F_y=x+z-2\lambda=0 \rightarrow \lambda=\frac{1}{2}x+\frac{1}{2}z \qquad \frac{4}{3}y=\frac{1}{2}x+\frac{1}{2}z \\ F_z=y+3\mu=0 \rightarrow \mu=-\frac{1}{3}y \rightarrow y=\frac{3}{8}x+\frac{3}{8}z \\ F_\lambda=-x-2y+6=0 \\ F_\mu=-x+3z=0 \end{cases}$$

$$\begin{cases} -x-2\left(\frac{3}{8}x+\frac{3}{8}z\right)+6=0 \\ -x+3z=0 \end{cases}$$

$$\begin{cases} -\frac{7}{4}x-\frac{3}{4}z+6=0 \\ x=3z \end{cases} \quad -\frac{7}{4}(3z)-\frac{3}{4}z=-6 \rightarrow$$

$21z+3z=24 \rightarrow z=1$, $x=3$, $y=\frac{3}{2}$, $\mu=-\frac{1}{2}$, $\lambda=2$, and maximum value

$f(3,\frac{3}{2},1)=\frac{9}{2}+\frac{3}{2}=6$.

28.) Minimize $f(x,y,z)=x^2+y^2+z^2$

Constraint: $x+y+z=120 \rightarrow x+y+z-120=0$;

$$F(x,y,z)=x^2+y^2+z^2-\lambda(x+y+z-120)$$
$$=x^2+y^2+z^2-\lambda x-\lambda y-\lambda z+120\lambda$$

$$\begin{cases} F_x = 2x - \lambda = 0 \to \lambda = 2x \\ F_y = 2y - \lambda = 0 \to \lambda = 2y \\ F_z = 2z - \lambda = 0 \to \lambda = 2z \\ F_\lambda = -x - y - z + 120 = 0 \end{cases}$$

$2x = 2y \to x = y$

$2y = 2z \to z = y = x$

$\to -x - (x) - (x) + 120 = 0 \to -3x + 120 = 0 \to$

$x = 40$, $y = 40$, $z = 40$, $\lambda = 80$, and

minimum value $f(40, 40, 40) = 1600 + 1600 + 1600 = 4800$

33.) minimize $f(x,y,z) = (x-2)^2 + (y-1)^2 + (z-1)^2$

Constraint: $x + y + z = 1 \to x + y + z - 1 = 0$;

$F(x,y,z,\lambda) = (x-2)^2 + (y-1)^2 + (z-1)^2 - \lambda(x+y+z-1)$

$= (x-2)^2 + (y-1)^2 + (z-1)^2 - \lambda x - \lambda y - \lambda z + \lambda \to$

$F_x = 2(x-2) - \lambda = 0 \to \lambda = 2(x-2)$

$F_y = 2(y-1) - \lambda = 0 \to \lambda = 2(y-1)$

$F_z = 2(z-1) - \lambda = 0 \to \lambda = 2(z-1)$

$2(x-2) = 2(y-1)$

$2(y-1) = 2(z-1)$

$F_\lambda = -x - y - z + 1 = 0 \leftarrow \begin{cases} x - 2 = y - 1 \to y = x - 1 \\ y - 1 = z - 1 \to z = y = x - 1 \end{cases}$

$\to -x - (x-1) - (x-1) + 1 = 0$

$\to -x - x + 1 - x + 1 + 1 = 0 \to -3x + 3 = 0 \to x = 1$,

$y = 0$, $z = 0$, $\lambda = -2$, and minimum

value $f(1,0,0) = 1 + 1 + 1 = 3$ so <u>minimum distance</u> is $\sqrt{3}$

Worksheet 6

1.) Let $F(x,y,z,\lambda,\mu) = x^2+y^2+z^2-\lambda(x-y+z)-\mu(3+x-2y+z)$ then

$F_x = 2x-\lambda-\mu=0$

$F_y = 2y+\lambda+2\mu=0$

$F_z = 2z-\lambda-\mu=0$

$F_\lambda = -x+y-z=0$

$F_\mu = -3-x+2y-z=0$

$2x+2y+\mu=0 \rightarrow \mu=-2x-2y$

$2y+2z+\mu=0 \rightarrow \mu=-2y-2z$

$-2x-\cancel{2y} = -\cancel{2y}-2z \rightarrow x=z$

$-x+y-(x)=0$

$-3-x+2y-(x)=0$

$y=2x$

$-3-2x+2y=0 \rightarrow -3-y+2y=0$

$\rightarrow y=3$, $x=3/2$, $z=3/2$ and

$f\left(\frac{3}{2},3,\frac{3}{2}\right) = \frac{9}{4}+9+\frac{9}{4} = \frac{27}{2}$ is the minimum value.

2.) Let $F(x,y,z,\lambda,\mu) = 10-x^2-2y^2-3z^2-\lambda(x-y-5)-\mu(x+y-z-2)$ then

$F_x = -2x-\lambda-\mu=0$

$F_y = -4y+\lambda-\mu=0$

$-2x-4y-2\mu=0 \rightarrow \mu=-x-2y$

$F_z = -6z+\mu=0 \rightarrow \mu=6z \longrightarrow 6z=-x-2y$ or

$F_\lambda = -x+y+5=0$

$F_\mu = -x-y+z+2=0$

$x+2y+6z=0$

$-x+y \quad =-5$

$-x-y+z=-2$

$3y+6z=-5$

$y+7z=-2$

$-15z=1 \rightarrow z=\frac{-1}{15}$, $y=\frac{-23}{15}$,

$x=\frac{52}{15}$ and $f\left(\frac{52}{15},\frac{-23}{15},\frac{-1}{15}\right) = \frac{-101}{15}$ is the maximum value.

3.) Maximize temperature $T = X^2 - 6X + 9 + Y^2$
subject to $X^2 + Y^2 = 25$:

Let $F(x, y, \lambda) = (X^2 - 6X + 9 + Y^2) - \lambda\,(X^2 + Y^2 - 25)$ then

$$\left.\begin{aligned} F_X &= 2X - 6 - 2\lambda X = 0 \\ F_Y &= 2Y - 2\lambda Y = 0 \end{aligned}\right\} \begin{aligned} &\lambda = \frac{X-3}{X} \\ &2Y(1-\lambda) = 0 \rightarrow \underline{\underline{Y=0}} \text{ or } \underline{\underline{\lambda = 1}} \end{aligned}$$

$F_\lambda = -X^2 - Y^2 + 25 = 0$

$\hookrightarrow X^2 + Y^2 = 25$;

if $\lambda = 1$ then $1 = \frac{X-3}{X} \rightarrow X = X - 3$

$\rightarrow 0 = -3$ (impossible!) ;

if $Y = 0$ then $X^2 + (0)^2 = 25 \rightarrow X = \pm 5$;

at $(5,0)$ temperature $T = 4^\circ$ is the lowest

and

at $(-5,0)$ temperature $T = 64^\circ$ is the highest.