

Worksheet 10

1.) a.) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ so that

$$\begin{aligned} f(x) &= x^2 + 3 + \sin(x^2) \\ &= x^2 + 3 + \left[(x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \dots \right] \\ &= x^2 + 3 + x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots \\ &= 3 + 2x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \dots \end{aligned}$$

b.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ and

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ so that

$$\begin{aligned} f(x) &= e^{-x} - \cos \sqrt{x} \\ &= \left[1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \dots \right] \\ &\quad - \left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \frac{(\sqrt{x})^8}{8!} - \dots \right] \\ &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \dots \\ &\quad - 1 + \frac{x}{2} - \frac{x^2}{24} + \frac{x^3}{720} - \frac{x^4}{40,320} + \dots \\ &= -\frac{1}{2}x + \frac{11}{24}x^2 - \frac{119}{720}x^3 + \frac{1679}{40,320}x^4 + \dots \end{aligned}$$

c.) $f(x) = x^3 \cos(2x)$

$$= x^3 \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \right]$$

$$= x^3 \left[1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots \right]$$

$$= x^3 - 2x^5 + \frac{2}{3}x^7 - \frac{4}{45}x^9 + \dots$$

d.) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ so

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 \quad \text{and (integrate)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{then}$$

$$f(x) = (1+x+x^2) \ln(1+x)$$

$$= (1+x+x^2) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) \quad (\text{expand})$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$+ x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \dots$$

$$+ x^3 - \frac{x^4}{2} + \dots$$

$$= x + \frac{1}{2}x^2 + \frac{5}{6}x^3 - \frac{5}{12}x^4 + \dots$$

e.) $f(x) = \frac{x^5}{1-2x^3} = x^5 \cdot \frac{1}{1-(2x^3)}$

$$= x^5 \left[1 + (2x^3) + (2x^3)^2 + (2x^3)^3 + \dots \right]$$

$$= x^5 \left[1 + 2x^3 + 4x^6 + 8x^9 + \dots \right]$$

$$= x^5 + 2x^8 + 4x^{11} + 8x^{14} + \dots$$

$$f.) \quad f(x) = x + (x+4)^{1/2} \rightarrow f'(x) = 1 + \frac{1}{2}(x+4)^{-1/2},$$

$$f''(x) = -\frac{1}{4}(x+4)^{-3/2}, \quad f'''(x) = \frac{3}{8}(x+4)^{-5/2}, \dots,$$

$$a_n = \frac{f^{(n)}(-3)}{n!} \quad \text{so} \quad a_0 = \frac{f(-3)}{0!} = \frac{-2}{1} = -2,$$

$$a_1 = \frac{f'(-3)}{1!} = \frac{\frac{3}{2}}{1} = \frac{3}{2}, \quad a_2 = \frac{f''(-3)}{2!} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8},$$

$$a_3 = \frac{f'''(-3)}{3!} = \frac{\frac{3}{8}}{6} = \frac{1}{16}, \dots \quad \text{then}$$

$$f(x) = x + (x+4)^{1/2} = -2 + \frac{3}{2}(x+3) - \frac{1}{8}(x+3)^2 + \frac{1}{16}(x+3)^3 + \dots$$

$$g.) \quad f(x) = \tan\left(\frac{\pi}{4}x\right) \rightarrow f'(x) = \frac{\pi}{4} \sec^2\left(\frac{\pi}{4}x\right),$$

$$f''(x) = \frac{\pi}{2} \sec\left(\frac{\pi}{4}x\right) \cdot \sec\left(\frac{\pi}{4}x\right) \tan\left(\frac{\pi}{4}x\right) \cdot \frac{\pi}{4} = \frac{\pi^2}{8} \sec^2\left(\frac{\pi}{4}x\right) \tan\left(\frac{\pi}{4}x\right),$$

$$f'''(x) = \frac{\pi^2}{8} \sec^2\left(\frac{\pi}{4}x\right) \cdot \sec^2\left(\frac{\pi}{4}x\right) \cdot \frac{\pi}{4}$$

$$+ \frac{\pi^2}{8} \tan\left(\frac{\pi}{4}x\right) \cdot 2 \sec\left(\frac{\pi}{4}x\right) \cdot \sec\left(\frac{\pi}{4}x\right) \tan\left(\frac{\pi}{4}x\right) \cdot \frac{\pi}{4}$$

$$= \frac{\pi^3}{32} \sec^4\left(\frac{\pi}{4}x\right) + \frac{\pi^3}{16} \sec^2\left(\frac{\pi}{4}x\right) \tan^2\left(\frac{\pi}{4}x\right), \dots,$$

$$a_n = \frac{f^{(n)}(1)}{n!} \quad \text{so} \quad a_0 = \frac{f(1)}{0!} = \frac{\tan\left(\frac{\pi}{4}\right)}{1} = \frac{1}{1} = 1,$$

$$a_1 = \frac{f'(1)}{1!} = \frac{\frac{\pi}{4} \sec^2\left(\frac{\pi}{4}\right)}{1} = \frac{\pi}{4} \cdot 2 = \frac{\pi}{2},$$

$$a_2 = \frac{f''(1)}{2!} = \frac{\frac{\pi^2}{8} \sec^2\left(\frac{\pi}{4}\right) \tan\left(\frac{\pi}{4}\right)}{2} = \frac{\pi^2}{16} \cdot 2 \cdot 1 = \frac{\pi^2}{8},$$

$$a_3 = \frac{f'''(1)}{3!} = \frac{1}{6} \left[\frac{\pi^3}{32} \sec^4\left(\frac{\pi}{4}\right) + \frac{\pi^3}{16} \sec^2\left(\frac{\pi}{4}\right) \tan^2\left(\frac{\pi}{4}\right) \right]$$

$$= \frac{1}{6} \left[\frac{\pi^3}{32} \cdot 4 + \frac{\pi^3}{16} \cdot 2 \cdot 1 \right] = \frac{\pi^3}{24}, \dots \text{ then}$$

$$f(x) = \tan\left(\frac{\pi}{4}x\right) = 1 + \frac{\pi}{2}(x-1) + \frac{\pi^2}{8}(x-1)^2 + \frac{\pi^3}{24} \cdot (x-1)^3 + \dots$$

$$2.) \quad a.) \quad 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x$$

$$\begin{aligned} b.) \quad & x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} \\ & = x^2 \cdot \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right] = x^2 e^x \end{aligned}$$

$$c.) \quad 1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x}$$

$$\begin{aligned} d.) \quad & x^4 - x^6 + x^8 - x^{10} + x^{12} - \dots \\ & = x^4 \cdot [1 - x^2 + x^4 - x^6 + x^8 - \dots] \\ & = x^4 \cdot [1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \dots] \\ & = x^4 \cdot \frac{1}{1 - (-x^2)} = \frac{x^4}{1 + x^2} \end{aligned}$$