

Section 10.5 (cont'd.)

21.) (See problem 10.)

$$\begin{aligned}\int_0^1 e^{-x^2} dx &\approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\right) dx \\&= \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right) \Big|_0^1 \\&= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \approx 0.742857142857\end{aligned}$$

22.) (See problem 2.)

$$\begin{aligned}\int_{-\frac{1}{4}}^{\frac{1}{4}} \ln(x^2+1) dx &\approx \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3}\right) dx \\&= \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3}\right) dx \\&= \left(\frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{21}\right) \Big|_{-\frac{1}{4}}^{\frac{1}{4}} \\&= \left(\frac{(\frac{1}{4})^3}{3} - \frac{(\frac{1}{4})^5}{10} + \frac{(\frac{1}{4})^7}{21}\right) - \left(\frac{(-\frac{1}{4})^3}{3} - \frac{(-\frac{1}{4})^5}{10} + \frac{(-\frac{1}{4})^7}{21}\right) \\&\approx 0.010227167039\end{aligned}$$

23.) See p. 687 and formula for $(1+x)^k$ with $k = -\frac{1}{2}$:

$$\begin{aligned} \frac{1}{\sqrt{1+x}} &= 1 - \frac{1}{2}x + \frac{\frac{1}{2} \cdot \frac{3}{2}}{2!} x^2 - \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{3!} x^3 \\ &+ \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}}{4!} x^4 - \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2}}{5!} x^5 + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} \cdot \frac{11}{2}}{6!} x^6 + \dots \end{aligned}$$

$$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots \quad \text{so}$$

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \dots \quad \text{and}$$

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 \quad \text{so}$$

$$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1+x^2}} dx \approx \int_0^{\frac{1}{2}} \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 \right) dx$$

$$= \left(x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 \right) \Big|_0^{\frac{1}{2}}$$

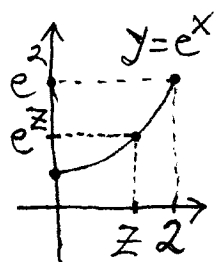
$$= \left(\frac{1}{2} \right) - \frac{1}{6} \left(\frac{1}{2} \right)^3 + \frac{3}{40} \left(\frac{1}{2} \right)^5 - \frac{5}{112} \left(\frac{1}{2} \right)^7$$

$$\approx 0.481161644345$$

25.) $f(x) = e^x, c = 1, [0, 2] \rightarrow$

$f'(x) = e^x = f''(x) = f'''(x) = \dots$ so $f^{(n)}(x) = e^x$ and

$$|f(x) - P_n(x)| = |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1} \right|$$



$$= \frac{e^z}{(n+1)!} |x-1|^{n+1}$$

where z is between 1 and x

and $0 \leq x \leq 2$

so $0 \leq z \leq 2$

$$\leq \frac{e^2}{(n+1)!} (1)^{n+1}$$

$$\leq \frac{3^2}{(n+1)!} = \frac{9}{(n+1)!} \leq .001 \text{ so}$$

$n+1 = 8$ works $\rightarrow \boxed{n=7}$

26.) $f(x) = \frac{1}{x}, c = 1, [1, \frac{3}{2}] \rightarrow$

$$f'(x) = -\frac{1}{x^2}, f''(x) = \frac{+2}{x^3}, f'''(x) = -\frac{3 \cdot 2}{x^4}, f^{(4)}(x) = \frac{+4 \cdot 3 \cdot 2}{x^5},$$

$$\dots, f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}} \text{ and}$$

$$|f(x) - P_n(x)| = |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1} \right|$$

$$= \left| \frac{\frac{(-1)^{n+1} (n+1)!}{z^{n+2}}}{(n+1)!} (x-1)^{n+1} \right|$$

$$= \frac{(n+1)!}{|z|^{n+2}} \cdot \frac{1}{(n+1)!} |x-1|^{n+1} \quad (\text{where } z \text{ is between } x \text{ and } c=1 \text{ and } 1 \leq x \leq \frac{3}{2})$$

$$= \frac{|x-1|^{n+1}}{|z|^{n+2}} \leq \frac{\left|\frac{3}{2}-1\right|^{n+1}}{(1)^{n+2}}$$

$$= \left(\frac{1}{2}\right)^{n+1} \leq 0.001 \rightarrow \ln\left(\frac{1}{2}\right)^{n+1} \leq \ln(0.001)$$

$$\rightarrow (n+1) \ln\left(\frac{1}{2}\right) \leq \ln(0.001)$$

$$\rightarrow n+1 \geq \frac{\ln(0.001)}{\ln\left(\frac{1}{2}\right)} \quad (\text{since } \ln\left(\frac{1}{2}\right) < 0)$$

$$\rightarrow n+1 \geq 9.965784 \rightarrow n \geq 8.965784$$

$$\rightarrow \boxed{n \geq 9} \text{ works.}$$

$$27.) \quad f(x) = e^{-x}, c = 0, [0, 1] \rightarrow$$

$$f'(x) = -e^{-x}, f''(x) = e^{-x}, f'''(x) = -e^{-x}, \dots,$$

$$f^{(n)}(x) = \pm e^{-x} \quad \text{so}$$

$$|f(x) - P_5(x)| = |R_5(x)| = \left| \frac{f^{(5+1)}(z)}{(5+1)!} (x-c)^{5+1} \right|$$

$$= \frac{e^{-z}}{(5+1)!} |x-0|^{5+1} \quad \text{where } 0 \leq z \leq x$$

$$\text{and } 0 \leq x \leq 1$$

$$\leq \frac{e^{-0}}{6!} (1)^6$$

$$= \frac{1}{6!}$$

$$= \boxed{.001388888}.$$

28.) $f(x) = \frac{1}{x}$, $c = 1$, $[1, \frac{3}{2}] \rightarrow$

$$f'(x) = -\frac{1}{x^2}, f''(x) = \frac{2}{x^3}, f'''(x) = -\frac{3 \cdot 2}{x^4}, \dots,$$

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}} \quad \text{then}$$

$$|f(x) - P_5(x)| = |R_5(x)| = \left| \frac{f^{(5+1)}(z)}{(5+1)!} (x-1)^{5+1} \right|$$

$$= \frac{\frac{6!}{|z|^7}}{6!} \cdot |x-1|^6$$

where $1 \leq z \leq x$
and $1 \leq x \leq \frac{3}{2}$

$$= \frac{|x-1|^6}{|z|^7}$$

$$\leq \frac{\left|\frac{3}{2} - 1\right|^6}{(1)^7}$$

$$= \left(\frac{1}{2}\right)^6$$

$$= \boxed{.015625}$$

Worksheet 11

① (a) $a_n = \cos(n\pi) \rightarrow$

$$a_1 = \cos \pi = -1$$

$$a_2 = \cos 2\pi = +1$$

$$a_3 = \cos 3\pi = -1$$

$$a_4 = \cos 4\pi = +1$$

$$a_5 = \cos 5\pi = -1$$

$$\lim_{n \rightarrow \infty} \cos(n\pi)$$

does not exist

(b) $a_n = \cos(n\pi) \div n$

$$a_1 = \cos \pi \div 1 = -1$$

$$a_2 = \cos 2\pi \div 2 = +1/2$$

$$a_3 = \cos 3\pi \div 3 = -1/3$$

$$a_4 = \cos 4\pi \div 4 = +1/4$$

$$a_5 = \cos 5\pi \div 5 = -1/5$$

$$\lim_{n \rightarrow \infty} \frac{\cos(n\pi)}{n} = 0$$

(c) $a_n = n \cos(n\pi) \rightarrow$

$$a_1 = 1 \cdot \cos \pi = -1$$

$$a_2 = 2 \cos 2\pi = +2$$

$$a_3 = 3 \cos 3\pi = -3$$

$$a_4 = 4 \cos 4\pi = +4$$

$$a_5 = 5 \cos 5\pi = -5$$

$$\lim_{n \rightarrow \infty} n \cos(n\pi)$$

does not exist

(d) $a_n = \sin(n\pi) \rightarrow$

$$a_1 = \sin \pi = 0$$

$$a_2 = \sin 2\pi = 0$$

$$a_3 = \sin 3\pi = 0$$

$$a_4 = \sin 4\pi = 0$$

$$a_5 = \sin 5\pi = 0$$

$$\lim_{n \rightarrow \infty} \sin(n\pi) = 0$$

e. $a_n = \sin(\pi/n)$
 $\lim_{n \rightarrow \infty} \sin(\pi/n)$
 $= \sin(0) = 0$

$a_1 = \sin \pi = 0$
 $a_2 = \sin(\pi/2) = 1$
 $a_3 = \sin(\pi/3) = \sqrt{3}/2 \approx .866025403$
 $a_4 = \sin(\pi/4) = \sqrt{2}/2 \approx .707106781$
 $a_5 = \sin(\pi/5) = .587785252$

f. $a_n = n \sin(1/n)$
 $\lim_{n \rightarrow \infty} n \sin(1/n)$
 $= \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{(1/n)} = 1$
 (Since $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$)

$a_1 = 1 \cdot \sin 1 = .8414709$
 $a_2 = 2 \cdot \sin(1/2) = .958851$
 $a_3 = 3 \sin(1/3) = .981584$
 $a_4 = 4 \sin(1/4) = .9896158$
 $a_5 = 5 \sin(1/5) = .9933466$

g. $a_n = \tan(\frac{\pi}{4} + \frac{n\pi}{2})$
 $\lim_{n \rightarrow \infty} \tan(\frac{\pi}{4} + \frac{n\pi}{2})$
does not exist

$a_1 = \tan(\frac{3\pi}{4}) = -1$
 $a_2 = \tan(\frac{5\pi}{4}) = +1$
 $a_3 = \tan(\frac{7\pi}{4}) = -1$
 $a_4 = \tan(\frac{9\pi}{4}) = +1$
 $a_5 = \tan(\frac{11\pi}{4}) = -1$

2. (a) $\lim_{n \rightarrow \infty} \cos(n\pi) \neq 0$ (SEE 1. (a))

so $\sum_{n=1}^{\infty} \cos(n\pi)$ diverges by n th term test

(b) $\lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = \sin \frac{\pi}{2} = 1 \neq 0$

so $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right)$ diverges by n th term test

(c) $\sum_{n=1}^{\infty} \frac{\tan\left(\frac{(4n+1)\pi}{4}\right)}{n^3}$

$$= \frac{\tan\left(\frac{5\pi}{4}\right)}{1^3} + \frac{\tan\left(\frac{9\pi}{4}\right)}{2^3} + \frac{\tan\left(\frac{13\pi}{4}\right)}{3^3} + \frac{\tan\left(\frac{17\pi}{4}\right)}{4^3} + \dots$$

$$= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots$$

converges since this is a

p -series with $p = 3 > 1$.

(d) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$; apply ratio test :

$$\lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1) \cdot n^n}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

$$= e > 1$$

so series diverges.

$$\begin{aligned}
 e.) \quad & \sum_{n=1}^{\infty} \frac{1 - \cos(n\pi)}{n^2} \\
 &= \frac{1 - \cos \pi}{1^2} + \frac{1 - \cos 2\pi}{2^2} + \frac{1 - \cos 3\pi}{3^2} + \frac{1 - \cos 4\pi}{4^2} + \dots \\
 &= \frac{1 - (-1)}{1^2} + \frac{1 - (1)}{2^2} + \frac{1 - (-1)}{3^2} + \frac{1 - (-1)}{4^2} + \frac{1 - (1)}{5^2} + \dots \\
 &= \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \dots = 2 \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right)
 \end{aligned}$$

$\boxed{\leq} 2 \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \right) = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$
 which is a convergent p -series since $p = 2 > 1$; since $0 \leq \sum_{n=1}^{\infty} \frac{1 - \cos(n\pi)}{n^2} \leq 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$;
 thus, $\sum_{n=1}^{\infty} \frac{1 - \cos(n\pi)}{n^2}$ converges since its value is smaller.

$$(3.) \quad f(x) = \ln(1-x), \quad c=0, \quad \left[-\frac{1}{2}, 0\right] \rightarrow$$

$$f'(x) = \frac{-1}{1-x} = -(1-x)^{-1}, \quad f''(x) = -(1-x)^{-2},$$

$$f'''(x) = -2(1-x)^{-3}, \quad f^{(4)}(x) = -3 \cdot 2 (1-x)^{-4}, \quad \dots$$

$$f^{(n)}(x) = -(n-1)! (1-x)^{-n} = \frac{-(n-1)!}{(1-x)^n} \quad \text{and}$$

$$f^{(n+1)}(x) = -n! (1-x)^{-(n+1)} = \frac{-n!}{(1-x)^{n+1}} ; \text{ then}$$

the absolute error is

$$|f(x) - P_n(x)| = |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1} \right|$$

$$= \frac{\frac{n!}{|1-z|^{n+1}}}{(n+1)!} |x-0|^{n+1}$$

$$= \frac{1}{n+1} \cdot \frac{|x|^{n+1}}{|1-z|^{n+1}} \quad (x, z \text{ are in } [-\frac{1}{2}, 0])$$

$$\leq \frac{1}{n+1} \cdot \frac{\left|-\frac{1}{2}\right|^{n+1}}{|1-0|^{n+1}}$$

$$= \frac{1}{n+1} \frac{\left(\frac{1}{2}\right)^{n+1}}{(1)^{n+1}}$$

$$= \frac{1}{n+1} \left(\frac{1}{2}\right)^{n+1} \leq 0.001 ;$$

by calculator $\boxed{n=7}$ works .