Section C.4

1.) $\frac{dY}{dx} = kY \rightarrow Y = ce^{kx}$

$x=0, Y=1 \rightarrow 1 = c \cdot 1 \rightarrow c = 1 \rightarrow Y = e^{kx}$

$x=3, Y=2 \rightarrow 2 = e^{3k} \rightarrow \ln 2 = 3k \rightarrow$

$k = \frac{1}{3}\ln 2 = .23104906 \rightarrow$

$$Y = e^{.23104906x}$$

6.) $\frac{dY}{dx} = kY \rightarrow Y = ce^{kx}$

$x=1, Y=4 \rightarrow 4 = ce^{k} \rightarrow c = 4e^{-k}$

$x=2, Y=1 \rightarrow 1 = ce^{2k} \rightarrow 1 = 4e^{-k} \cdot e^{2k} \rightarrow$

$\frac{1}{4} = e^{k} \rightarrow k = \ln\frac{1}{4} = -1.386294361$ and

$c = 4e^{-k} = 16 \rightarrow$

$$Y = 16e^{-1.386294361x}$$

7.) $\frac{dA}{dt} = kA \rightarrow A = ce^{kt}$; then

if $t=0$, $A = \$2000 \rightarrow 2000 = ce^{0} = c \cdot 1 = c$

$\rightarrow A = 2000e^{kt}$; if $t=5$, $A = \$2983.65$

then $2983.65 = 2000e^{5k} \rightarrow$

$1.491825 = e^{5k} \rightarrow \ln 1.491825 = \ln e^{5k}$

$\rightarrow \ln 1.491825 = 5k \rightarrow$

$k \approx 0.0800000405$ so that

$$A = 2000\, e^{0.0800000405\, t}\ ;$$

if $t = 10$ yrs., then $A \approx \$4451.08$.

15.)

$$\frac{dN}{dt} = kN(500-N) \rightarrow \int \frac{1}{N(500-N)}\, dN = \int k\, dt \rightarrow$$

$$\int \left[\frac{A}{N} + \frac{B}{500-N}\right] dN = kt + c \rightarrow \int \left[\frac{\frac{1}{500}}{N} + \frac{\frac{1}{500}}{500-N}\right] dN = kt + c \rightarrow$$

$$\frac{1}{500} \ln N - \frac{1}{500} \ln(500-N) = kt + c \rightarrow$$

$$\ln N - \ln(500-N) = 500kt + 500c \rightarrow$$

$$\ln \frac{N}{500-N} = 500kt + c \rightarrow$$

$$\frac{N}{500-N} = e^{500kt+c} = e^{c} e^{500kt} = c\, e^{500kt} \rightarrow$$

$$\frac{N}{500-N} = c\, e^{500kt}\ ;$$ if $\underline{t=0}$, $\underline{N=100}$ then

$$\frac{100}{400} = c \cdot e^{0} \rightarrow c = \frac{1}{4} \rightarrow \frac{N}{500-N} = \frac{1}{4}\, e^{500kt}\ ;$$

if $\underline{t=4}$, $\underline{N=200}$ then

$$\frac{200}{300} = \frac{1}{4}\, e^{2000k} \rightarrow \frac{8}{3} = e^{2000k} \rightarrow$$

$$\ln \frac{8}{3} = \ln e^{2000k} = 2000k \rightarrow$$

$$k = \frac{1}{2000} \ln \frac{8}{3} = 0.000490414 \text{ (use later)} \rightarrow$$

$$\frac{N}{500-N} = \frac{1}{4} e^{500kt} \longrightarrow$$

$$N = 125 e^{500kt} - \frac{1}{4} e^{500kt} \cdot N \longrightarrow$$

$$N + \frac{1}{4} e^{500kt} N = 125 e^{500kt} \rightarrow$$

$$\left(1 + \frac{1}{4} e^{500kt}\right) N = 125 e^{500kt} \longrightarrow$$

$$N = \frac{125 e^{500kt}}{1 + \frac{1}{4} e^{500kt}} \longrightarrow$$

$$N = \frac{500}{4e^{-500kt} + 1} \longrightarrow \text{ (plug in k)}$$

$$\boxed{N = \frac{500}{4e^{-0.245207313\, t} + 1}} .$$

20.) $y = \frac{-1}{kt+c}$ and

$t=0,\ y=75 \rightarrow \quad 75 = \frac{-1}{c} \rightarrow \underbrace{c = \frac{-1}{75}}$, and

$t=1,\ y=12 \rightarrow \quad 12 = \dfrac{-1}{k + \frac{-1}{75}} = \dfrac{-75}{75k-1} \rightarrow$

$12(75k-1) = -75 \rightarrow \quad 900k = -63 \rightarrow$

$k = -.07 \quad \rightarrow \quad y = \dfrac{-1}{-.07t + \frac{-1}{75}}$ or

$$y = \frac{75}{5.25t + 1}$$

28.) $\dfrac{dQ}{dt}$ = (rate in) − (rate out)

$= \left(0 \dfrac{\text{lbs.}}{\text{gal}}\right)\left(\dfrac{5 \text{ gal.}}{\text{min.}}\right) - \left(\dfrac{Q \text{ lbs.}}{100 \text{ gal.}}\right)\left(\dfrac{5 \text{ gal.}}{\text{min.}}\right)$

$\rightarrow \quad \dfrac{dQ}{dt} = \dfrac{-Q}{20} \quad \rightarrow \quad \dfrac{dQ}{dt} = \left(\dfrac{-1}{20}\right) Q$

(rate is proportional to amount!)

$\rightarrow \quad Q = ce^{\frac{-1}{20}t}$; if $t=0$, $Q = 25$ lbs.

then $25 = ce^0 = c \cdot 1 = c \rightarrow$

a.) $Q = 25e^{\frac{-1}{20}t}$.

b.) If $Q = 15$ lbs., then $15 = 25e^{\frac{-1}{20}t} \rightarrow$

$\dfrac{3}{5} = e^{\frac{-1}{20}t} \rightarrow \ln(3/5) = \ln e^{\frac{-1}{20}t} \rightarrow$

$\ln(3/5) = \dfrac{-1}{20}t \rightarrow$

$t = -20 \cdot \ln(3/5) \approx$ 10.21 min.

29.) $\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$

$= \left(\frac{\frac{1}{2}\text{ lb.}}{\text{gal.}}\right)\left(\frac{5\text{ gal.}}{\text{min.}}\right) - \left(\frac{Q\text{ lbs.}}{100\text{ gal.}}\right)\left(\frac{5\text{ gal.}}{\text{min.}}\right) \rightarrow$

$\frac{dQ}{dt} = \frac{5}{2} - \frac{1}{20}Q \rightarrow \frac{dQ}{dt} + \frac{1}{20}Q = \frac{5}{2} \rightarrow$

(first-order linear) $\mu = e^{\int \frac{1}{20}dt} = e^{\frac{1}{20}t} \rightarrow$

$e^{\frac{1}{20}t} \cdot \frac{dQ}{dt} + \frac{1}{20}e^{\frac{1}{20}t} \cdot Q = \frac{5}{2}e^{\frac{1}{20}t} \rightarrow$

$D\left(e^{\frac{1}{20}t} \cdot Q\right) = \frac{5}{2}e^{\frac{1}{20}t} \rightarrow$

$e^{\frac{1}{20}t} \cdot Q = \int \frac{5}{2}e^{\frac{1}{20}t}dt = \frac{5}{2} \cdot 20e^{\frac{1}{20}t} + c \rightarrow$

$e^{\frac{1}{20}t} \cdot Q = 50e^{\frac{1}{20}t} + c \rightarrow \underline{Q = 50 + ce^{-\frac{1}{20}t}}$;

$t = 0,\ Q = 0 \rightarrow 0 = 50 + ce^{0} = 50 + c \rightarrow c = -50$

$\rightarrow \boxed{Q = 50 - 50e^{-\frac{1}{20}t}}$;

if $t = 30$ min., then

$Q = 50 - 50e^{-\frac{1}{20}(30)} \approx$ 38.84 lbs.

30.) $\frac{ds}{dh} = \frac{k}{h} \rightarrow \int ds = \int \frac{k}{h}\, dh \rightarrow$

$S = k \ln h + c$ and

$h = 2,\ S = 25 \rightarrow \quad 25 = k \ln 2 + c$

$h = 6,\ S = 12 \rightarrow \quad 12 = k \ln 6 + c$

(subtract) $\quad 13 = k \ln 2 - k \ln 6 \rightarrow$

$13 = k(\ln 2 - \ln 6) = k \ln \frac{2}{6} = k \ln 1/3 \rightarrow$

$k = \frac{13}{\ln 1/3} = \underline{-11.83310995}$; so

$c = 25 - k \ln 2 = 33.2020868 \rightarrow$

$$\boxed{S \approx -11.83310995 \ln h + 33.2020868}$$

31.) Let M be amount of moisture (given in ounces, for example) at time t ; then

$\frac{dM}{dt} = kM \rightarrow M = ce^{kt}$

c ← initial amount of moisture ;

if $t = 1$, then $M = 60\%$ of $c = 0.6c \rightarrow$

$0.6\cancel{c} = \cancel{c}e^k \rightarrow 0.6 = e^k \rightarrow \ln(0.6) = \ln e^k$

$\rightarrow k = \ln(0.6) \rightarrow M = ce^{(\ln 0.6)t} \rightarrow$

$M = c\left(e^{\ln(0.6)}\right)^t = c(0.6)^t \rightarrow \boxed{M = c(0.6)^t}$;

if $M = 20\%$ of $c = 0.2c$, then

$0.2\cancel{c} = \cancel{c}(0.6)^t \rightarrow \ln(0.2) = \ln(0.6)^t \rightarrow$

$\ln(0.2) = t \cdot \ln(0.6) \rightarrow$

$t = \dfrac{\ln(0.2)}{\ln(0.6)} \approx$ 3.15 hrs.

32.) $\frac{1}{x} \cdot \frac{dx}{dt} = \frac{1}{y} \frac{dy}{dt} \rightarrow \int \frac{1}{x} dx = \int \frac{1}{y} dy \rightarrow$

$\ln x = \ln y + c \rightarrow e^{\ln x} = e^{\ln y + c} \rightarrow$

$x = e^c e^{\ln y} = cy \rightarrow x = cy \rightarrow y = \frac{1}{c} x$

$\rightarrow y = cx$.

33.) $\dfrac{dP}{dt} = kP + N \longrightarrow P' - kP = N$

(this is first-order linear!)

$e^{\int -k\,dt} = e^{-kt} \rightarrow$

$e^{-kt} P' - k e^{-kt} P = N e^{-kt} \rightarrow$

$D(e^{-kt} P) = N e^{-kt} \rightarrow$

$e^{-kt} P = \int N e^{-kt} dt = N \cdot \frac{1}{-k} e^{-kt} + c \rightarrow$

$$P = \frac{-N}{k} + c e^{kt}$$

Q3 : Determine the maximum value

of $S = 1000 - 4t - 0.000174413\,(500-2t)^{5/2}$:

$$S' = -4 + 0.000872065\,(500-2t)^{3/2} = 0 \rightarrow$$

$$(500-2t)^{3/2} = 4586.81 \rightarrow$$

$$500 - 2t = 4586.81^{2/3} \rightarrow$$

$$500 - 2t = 276.1 \rightarrow$$

$t = 112$ min. and

maximum amount is

$S = 331.3$ lbs.

Math 16C
Vogler
Worksheet 1

Let S represent the amount (in pounds) of salt in each tank at time t minutes. Find a formula for S for each of the following and then answer the particular questions.

1.) A solution containing 1/2 lb. of salt per gallon flows into a tank at the rate of 2 gal./min. and the well-stirred mixture flows out of the tank at the same rate. The tank initially holds 100 gallons of solution containing 5 lbs. of salt.

a.) How much salt is in the tank after 30 minutes ?

b.) How much salt do you expect to be in the tank as t gets infinitely large ?

2.) A solution containing 1 lb. of salt per gallon flows into a tank at the rate of 5 gal./min. and the well-stirred mixture flows out of the tank at the rate of 4 gal./min. The tank initially holds 50 gallons of water containing no salt.

a.) How many gallons of solution are in the tank after 1 hour ?

b.) How much salt is in the tank after 1 hour ?

c.) Assuming that the tank is very large, how much salt per gallon do you expect to be in the tank as t get infinitely large ?

Worksheet 1

(1.) $\frac{dS}{dt}$ = (rate in) − (rate out)

$= \left(\frac{1}{2}\ \frac{\text{lb.}}{\text{gal.}}\right)\left(2\ \frac{\text{gal.}}{\text{min.}}\right) - \left(\frac{S\ \text{lb.}}{100\ \text{gal.}}\right)\left(2\ \frac{\text{gal.}}{\text{min.}}\right)$

$= 1 - \frac{1}{50}S \rightarrow \frac{dS}{dt} + \frac{1}{50}S = 1 \rightarrow$

$e^{\int \frac{1}{50}\,dt} = e^{\frac{1}{50}t} \rightarrow e^{\frac{1}{50}t}S' + \frac{1}{50}e^{\frac{1}{50}t}S = e^{\frac{1}{50}t} \rightarrow$

$D\left(e^{\frac{1}{50}t}S\right) = e^{\frac{1}{50}t} \rightarrow e^{\frac{1}{50}t}S = \int e^{\frac{1}{50}t}\,dt \rightarrow$

$e^{\frac{1}{50}t}S = 50e^{\frac{1}{50}t} + c \rightarrow$

$\boxed{S = 50 + ce^{-\frac{1}{50}t}}$; and

$t=0,\ S = 5$ lbs. $\rightarrow 5 = 50 + ce^{0} \rightarrow c = -45 \rightarrow$

$\boxed{S = 50 - 45e^{\frac{-1}{50}t}}$.

(a.) $t = 30$ min. $\rightarrow S = \boxed{25.3 \text{ lbs.}}$

(b.) $\lim_{t\to\infty} S = 50 - 45\cdot(0) = \boxed{50 \text{ lbs.}}$

(2.) $\frac{dS}{dt}$ = (rate in) − (rate out)

$$= \left(1\ \frac{\text{lb.}}{\text{gal.}}\right)\left(5\ \frac{\text{gal.}}{\text{min.}}\right) - \left(\frac{S\ \text{lb.}}{50+t\ \text{gal.}}\right)\left(4\ \frac{\text{gal.}}{\text{min.}}\right)$$

$$= 5 - \frac{4}{50+t} S$$ or

$$\frac{dS}{dt} + \frac{4}{50+t} S = 5$$; then

$$e^{\int \frac{4}{50+t}\,dt} = e^{4 \ln(50+t)} = e^{\ln(50+t)^4}$$

$$= (50+t)^4 \longrightarrow$$

$$(50+t)^4 S' + 4(50+t)^3 S = 5(50+t)^4 \longrightarrow$$

$$D\{(50+t)^4 S\} = 5(50+t)^4 \longrightarrow$$

$$(50+t)^4 S = \int 5(50+t)^4\,dt = (50+t)^5 + c \longrightarrow$$

$$S = (50+t) + \frac{c}{(50+t)^4}$$; and

$t=0$, $S = 0$ lb. $\longrightarrow$

$$0 = 50 + \frac{c}{50^4} \longrightarrow c = -50^5 \longrightarrow$$

$$\boxed{S = 50 + t - \frac{50^5}{(50+t)^4}}.$$

a.) $50 + 60 = 110$ gal.

b.) $t = 60$ min. $\rightarrow$ $S = 107.87$ lbs.

c.) salt per gallon is given by

$$\lim_{t\to\infty} \frac{50 + t - \frac{50^5}{(50+t)^4}}{50+t} \quad \frac{\text{lbs.}}{\text{gal.}}$$

$$= \lim_{t\to\infty} \left[1 - \frac{50^5}{(50+t)^5} \right] = 1 \text{ lb./gal.}$$