Section 7.5

2.) $z = x^2 + y^2 + 2x - 6y + 6 \rightarrow$

$z_x = 2x + 2 = 0 \rightarrow x = -1$,

$z_y = 2y - 6 = 0 \rightarrow y = 3$; then

$z_{xx} = 2$, $z_{yy} = 2$, $z_{xy} = 0$ and

For $\boxed{(-1,3)}$: $D = z_{xx} z_{yy} - (z_{xy})^2 = (2)(2) - (0)^2 = 4 > 0$

and $z_{xx} = 2 > 0$ so $(-1,3)$ determines a minimum value of $z = -4$.

3.) $z = \sqrt{x^2 + y^2 + 1} \longrightarrow$

$z_x = \frac{1}{2}(x^2+y^2+1)^{-1/2} \cdot (2x) = \dfrac{x}{(x^2+y^2+1)^{1/2}} = 0 \rightarrow$

$\boxed{x=0}$; $z_y = \frac{1}{2}(x^2+y^2+1)^{-1/2} \cdot (2y) = \dfrac{y}{(x^2+y^2+1)^{1/2}} = 0 \rightarrow$

$\boxed{y=0}$;

$$z_{xx} = \frac{(x^2+y^2+1)^{1/2}(1) - x \cdot \frac{1}{2}(x^2+y^2+1)^{-1/2}(2x)}{((x^2+y^2+1)^{1/2})^2},$$

$$z_{yy} = \frac{(x^2+y^2+1)^{1/2}(1) - y \cdot \frac{1}{2}(x^2+y^2+1)^{-1/2} \cdot (2y)}{((x^2+y^2+1)^{1/2})^2},$$

$$z_{xy} = \frac{(x^2+y^2+1)^{1/2}(0) - x \cdot \frac{1}{2}(x^2+y^2+1)^{-1/2} \cdot 2y}{((x^2+y^2+1)^{1/2})^2};$$

$z_{xx}(0,0) = \dfrac{1-0}{1} = 1$, $z_{yy}(0,0) = \dfrac{1-0}{1} = 1$,

$z_{xy}(0,0) = \dfrac{0}{1} = 0$, then

For $\boxed{(0,0)}$: $D = z_{xx} \cdot z_{yy} - (z_{xy})^2 = (1)(1) - (0)^2 = 1 > 0$

and $z_{xx} = 1 > 0$ so $(0,0)$ determines a minimum value of $z = 1$.

5.) $z = (x-1)^2 + (y-3)^2 \longrightarrow$

$z_x = 2(x-1) = 0 \rightarrow \boxed{x=1}$
$z_y = 2(y-3) = 0 \rightarrow \boxed{y=3}$ $\Big\}$ $\boxed{(1,3)}$

$z_{xx} = 2, \quad z_{yy} = 2, \quad z_{xy} = 0 \rightarrow$

For $\boxed{(1,3)}$:

$D = z_{xx} z_{yy} - (z_{xy})^2 = (2)(2) - (0)^2 = 4 > 0$

and $z_{xx} = 2 > 0$ so $(1,3)$ determines a minimum value of $z = 0$.

7.) $z = 2x^2 + 2xy + y^2 + 2x - 3 \longrightarrow$

$z_x = 4x + 2y + 2 = 0 \rightarrow \boxed{y = -2x - 1}$
$z_y = 2x + 2y = 0 \rightarrow \boxed{y = -x}$ $\Big\}$

$\longrightarrow -x = -2x - 1 \rightarrow x = -1$ and $y = 1 \rightarrow \boxed{(-1,1)}$:

$z_{xx} = 4, \quad z_{yy} = 2, \quad z_{xy} = 2$ so

For $\boxed{(-1,1)}$:

$D = z_{xx} z_{yy} - (z_{xy})^2 = (4)(2) - (2)^2 = 4 > 0$

and $z_{xx} = 4 > 0$ so $(-1,1)$ determines a minimum value of $z = -4$.

9.) $z = -5x^2 + 4xy - y^2 + 16x + 10 \longrightarrow$

$z_x = -10x + 4y + 16 = 0 \rightarrow y = -4 + \frac{5}{2}x$
$z_y = 4x - 2y = 0 \longrightarrow y = 2x$ $\Big\}$

$2x = -4 + \frac{5}{2}x \rightarrow 4 = \frac{1}{2}x \rightarrow x = 8$ and $y = 16$

For $\boxed{(8,16)}$:

$z_{xx} = -10, \quad z_{yy} = -2, \quad z_{xy} = 4 \longrightarrow$

For $\boxed{(8,16)}$:

$D = z_{XX} \cdot z_{YY} - (z_{XY})^2 = (-10)(-2) - (4)^2 = 4 > 0$ and $z_{XX} = -10$ so $(8,16)$ determines a maximum value of $z = 74$.

10.) $z = x^2 + 6XY + 10Y^2 - 4Y + 4 \longrightarrow$

$z_X = 2X + 6Y = 2(X+3Y) = 0 \rightarrow \boxed{X+3Y=0}$,

$z_Y = 6X + 20Y - 4 = 2(3X+10Y-2) = 0 \rightarrow$

$\boxed{3X+10Y-2=0}$; $X = -3Y \xrightarrow{\text{sub}}$

$3(-3Y) + 10Y - 2 = 0 \rightarrow -9Y + 10Y - 2 = 0 \rightarrow$

$Y=2$ and $X=-6$;

$z_{XX} = 2$, $z_{YY} = 20$, $z_{XY} = 6$ then

For $\boxed{(-6,2)}$: $D = z_{XX} \cdot z_{YY} - (z_{XY})^2 = (2)(20) - (6)^2 = 4 > 0$

and $z_{XX} = 2 > 0$ so $(-6,2)$ determines a minimum value of $z = 0$.

14.) $z = x^2 - 3XY - Y^2 \longrightarrow$

$z_X = 2X - 3Y = 0 \rightarrow \boxed{X = \frac{3}{2}Y}$,

$z_Y = -3X - 2Y = 0 \rightarrow \boxed{X = \frac{-2}{3}Y}$; then

$\frac{3}{2}Y = \frac{-2}{3}Y \longrightarrow \frac{5}{6}Y = 0 \longrightarrow Y=0,\ X=0$,

$z_{XX} = 2$, $z_{YY} = -2$, $z_{XY} = -3$

For $\boxed{(0,0)}$: $D = z_{XX} \cdot z_{YY} - (z_{XY})^2 = (2)(-2) - (-3)^2 = -13 < 0$

so $(0,0)$ determines a saddle point at $z = 0$.

18.) $z = e^{-(x^2+y^2)} \longrightarrow$

$$z_x = -2x\,e^{-(x^2+y^2)} = 0 \to x = 0$$
$$z_y = -2y\,e^{-(x^2+y^2)} = 0 \to y = 0$$

so critical point is $\boxed{(0,0)}$;

$$z_{xx} = 4x^2 e^{-(x^2+y^2)} - 2e^{-(x^2+y^2)} = (4x^2-2)e^{-(x^2+y^2)},$$
$$z_{yy} = 4y^2 e^{-(x^2+y^2)} - 2e^{-(x^2+y^2)} = (4y^2-2)e^{-(x^2+y^2)},$$
$$z_{xy} = 4xy\,e^{-(x^2+y^2)} \longrightarrow$$

For (0,0) : $D = z_{xx}z_{yy} - (z_{xy})^2 = (-2)(-2) - (0)^2 = 4 > 0$ and $z_{xx} = -2 < 0$ so $(0,0)$ determines a maximum value of $z = 1$.

20.) $z = \dfrac{-4x}{x^2+y^2+1} \longrightarrow$

$$z_x = \frac{(x^2+y^2+1)(-4) - (-4x)\cdot(2x)}{(x^2+y^2+1)^2}$$
$$= \frac{-4x^2-4y^2-4+8x^2}{(x^2+y^2+1)^2} = \frac{4x^2-4y^2-4}{(x^2+y^2+1)^2},$$
$$z_y = \frac{(x^2+y^2+1)(0) - (-4x)(2y)}{(x^2+y^2+1)^2} = \frac{8xy}{(x^2+y^2+1)^2},$$
$$z_{xx} = \frac{(x^2+y^2+1)^2\cdot(8x) - (4x^2-4y^2-4)\cdot 2(x^2+y^2+1)(2x)}{(x^2+y^2+1)^4}$$
$$= \frac{8x\cancel{(x^2+y^2+1)}\cdot[(x^2+y^2+1) - (2x^2-2y^2-2)]}{(x^2+y^2+1)^{\cancel{4}\,3}}$$

$$= \frac{8X(3-X^2+3Y^2)}{(X^2+Y^2+1)^3} \quad ,$$

$$z_{YY} = \frac{(X^2+Y^2+1)^2 \cdot (8X) - 8XY \cdot 2(X^2+Y^2+1)\cdot 2Y}{(X^2+Y^2+1)^4}$$

$$= \frac{8X(\cancel{X^2+Y^2+1})\left[(X^2+Y^2+1)-4Y^2\right]}{(X^2+Y^2+1)^{\cancel{4}\,3}}$$

$$= \frac{8X(1+X^2-3Y^2)}{(X^2+Y^2+1)^3} \quad ,$$

$$z_{XY} = z_{YX} = \frac{(X^2+Y^2+1)^2(8Y) - 8XY\cdot 2(X^2+Y^2+1)\cdot 2X}{(X^2+Y^2+1)^4}$$

$$= \frac{8Y(\cancel{X^2+Y^2+1})\left[(X^2+Y^2+1)-4X^2\right]}{(X^2+Y^2+1)^{\cancel{4}\,3}}$$

$$= \frac{8Y(1+Y^2-3X^2)}{(X^2+Y^2+1)^3} \quad ;$$

<u>critical points</u>:

$z_X = 0 \to 4X^2-4Y^2-4=0 \to \boxed{X^2-Y^2-1=0}$;

$z_Y = 0 \to 8XY = 0 \to \boxed{X=0 \text{ or } Y=0}$; then

$X=0 \to -Y^2-1=0 \to -(Y^2+1)=0$ (impossible) ;

$Y=0 \to X^2-1 = (X-1)(X+1)=0 \to X=1$ or $X=-1$

so critical pts. are $X=1, Y=0$ and $X=-1, Y=0$;

<u>For (1,0)</u>: $D = z_{XX}\cdot z_{YY} - (z_{XY})^2 = (2)(2)-(0)^2 = 4>0$ and $z_{XX}>0$ so $(1,0)$ determines a <u>minimum value</u> of $z=-2$;

<u>For (-1,0)</u>: $D = z_{XX}\cdot z_{YY} - (z_{XY})^2 = (-2)(-2)-(0)^2 = 4>0$

and $z_{xx} < 0$ so $(-1,0)$ determines a <u>maximum value</u> of $z = 2$

Worksheet 4

1. a.) $z = \sin 3x + \cos 5y \longrightarrow$

$z_x = \cos 3x \cdot 3$, $z_y = -\sin 5y \cdot 5$

b.) $z = \cos(x^2 y) \longrightarrow$

$z_x = -\sin(x^2 y) \cdot 2xy$,

$z_y = -\sin(x^2 y) \cdot x^2$

c.) $z = \ln(\sec x - \tan y) \longrightarrow$

$z_x = \dfrac{1}{\sec x - \tan y} \cdot \sec x \tan x$,

$z_y = \dfrac{1}{\sec x - \tan y} \cdot -\sec^2 y$

d.) $z = e^{x^2 \sin y} \longrightarrow$

$z_x = e^{x^2 \sin y} \cdot 2x \sin y$,

$z_y = e^{x^2 \sin y} \cdot x^2 \cos y$

e.) $z = \dfrac{\ln y}{\ln x + \ln y} \longrightarrow$

$z_x = \dfrac{(\ln x + \ln y)(0) - \ln y \left(\frac{1}{x}\right)}{(\ln x + \ln y)^2}$,

$z_y = \dfrac{(\ln x + \ln y)\left(\frac{1}{y}\right) - \ln y \left(\frac{1}{y}\right)}{(\ln x + \ln y)^2}$

f.) $z = \cot^2(4Y) - \csc^3(5XY) \rightarrow$

$z_X = 0 - 3\csc^2(5XY)\cdot(-1)\csc(5XY)\cot(5XY)\cdot\{5Y\}$,

$z_Y = 2\cot(4Y)\cdot(-1)\csc^2(4Y)\cdot\{4\} - 3\csc^2(5XY)\cdot(-1)\csc(5XY)\cot(5XY)\cdot\{5X\}$.

g.) $z = \{7Y^2 + 6\cot(11 - \pi e^{-\frac{1}{2}Y^{1/2}})\}^{1/2} \rightarrow z_X = 0!$,

$z_Y = \frac{1}{2}\{7Y^2 + 6\cot(11 - \pi\cdot e^{-\frac{1}{2}Y^{1/2}})\}^{-1/2}\cdot[14Y +$

$-6\csc^2(11 - \pi e^{-\frac{1}{2}Y^{1/2}})\cdot\{-\pi\cdot e^{-\frac{1}{2}Y^{1/2}}\cdot\frac{-1}{2}\cdot\frac{1}{2}Y^{-1/2}\}]$

h.) $z = \ln(\ln(\ln(3X - 9Y^2))) \rightarrow$

$$z_X = \frac{1}{\ln(\ln(3X-9Y^2))}\cdot\frac{1}{\ln(3X-9Y^2)}\cdot\frac{1}{3X-9Y^2}\cdot(3) ,$$

$$z_Y = \frac{1}{\ln(\ln(3X-9Y^2))}\cdot\frac{1}{\ln(3X-9Y^2)}\cdot\frac{1}{3X-9Y^2}\cdot(-18Y)$$

2.) $z = \sin(XY) \rightarrow$

$z_X = \cos(XY)\cdot Y$,

$z_Y = \cos(XY)\cdot X$,

$z_{XX} = -Y\sin(XY)\cdot Y$,

$z_{YY} = -X\sin(XY)\cdot X$,

$z_{XY} = \cos(XY)\cdot 1 + Y\cdot -\sin(XY)\cdot X$,

$z_{YX} = \cos(XY)\cdot 1 + X\cdot -\sin(XY)\cdot Y$.

3.) $z = (y+7x)^2 \longrightarrow$

$z_x = 2(y+7x)\cdot 7 = 14y + 98x$, $z_{xx} = 98$ and

$z_y = 2(y+7x)\cdot 1 = 2y + 14x$, $z_{yy} = 2$ then

$$z_{xx} = 98 = 49 \cdot 2 = 49 \cdot z_{yy} .$$

4.) $z_x = y^3 - 2x + 5$ so "partial anti-derivative" is

$$z = xy^3 - x^2 + 5x + c(y) ,$$ where

$c(y)$ represents a function of y. Now take the y-partial derivative of z getting

$$z_y = 3xy^2 + c'(y) = 3xy^2 + 2ye^{y^2} .$$

Then $c'(y) = 2ye^{y^2} \rightarrow$

$c(y) = e^{y^2} + c_1$, where c_1 is any constant. Thus,

$$\boxed{z = xy^3 - x^2 + 5x + e^{y^2} + c_1} .$$