1.

(a) State the PRECISE definition of the limit for

$$\lim_{x\to a} f(x) = L$$

– Given any number $\varepsilon > 0$, there exists another number $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$

or

– $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

(b) Give an ε, δ-proof for the following limit. This is a writing exercise.

$$\lim_{x\to -2} (x^2+3) = 7$$

$\varepsilon > 0$ is given. Need to find $\delta > 0$ s.t.

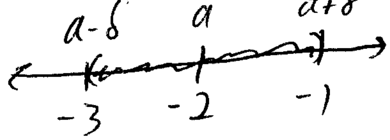
$|x-(-2)| = |x+2| < \delta \Rightarrow |f(x) - 7| < \varepsilon$

Begin w/ $|f(x) - 7| < \varepsilon$ & solve for $|x+2| < \delta$

$|f(x) - 7| < \varepsilon \iff |x^2 + 3 - 7| < \varepsilon \iff |x^2 - 4| < \varepsilon$

$\iff |x-2||x+2| < \varepsilon$. Want to remove $|x-2|$

Assume $\delta \le 1$,



$\Rightarrow -3 < x < -1 \Rightarrow -5 < x-2 < -3 \Rightarrow 3 < |x-2| < 5$.

Then $|x-2||x+2| < 5|x+2| < \varepsilon \iff |x+2| < \frac{\varepsilon}{5}$

Choose $\delta = \min\{\frac{\varepsilon}{5}, 1\}$.

Hence, if $|x+2| < \delta$, it follows that $|f(x) - 7| < \varepsilon$

2. Determine the following limits

(a) $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^2-1}$ $= \text{`}\dfrac{0}{0}\text{'} = \lim\limits_{x\to 1} \dfrac{(x+2)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \dfrac{3}{2}$

(b) $\lim\limits_{h\to 0} \dfrac{\sin 5h}{2h}$ $= \text{`}\dfrac{0}{0}\text{'} = \dfrac{5}{2} \cdot \lim\limits_{h\to 0} \underbrace{\dfrac{\sin \cancel{5} h}{\cancel{5} h}}_{1} = \dfrac{5}{2}$

(c) $\lim\limits_{x\to 1^-} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$ $= \text{`}\dfrac{0}{0}\text{'} = \lim\limits_{x\to 1^-} \dfrac{\sqrt{2x}\,\cancel{(x-1)}}{-\cancel{(x-1)}} = -\sqrt{2}$

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(d) $\lim\limits_{x\to 4} \dfrac{\sqrt{5+x}-3}{x-4}$ $= \text{`}\dfrac{0}{0}\text{'} = \lim\limits_{x\to 4} \dfrac{\sqrt{5+x}-3}{x-4} \cdot \dfrac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$= \lim\limits_{x\to 4} \dfrac{x+5-9}{(x-4)(\sqrt{x+5}+3)} = \lim\limits_{x\to 4} \dfrac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x+5}+3)} = \dfrac{1}{6}$

(e) $\lim\limits_{x\to\infty} \dfrac{3+\cos(x)}{x^4}$ (HINT : Use the Squeeze Principle.)

$-1 \le \cos x \le 1 \Rightarrow 2 \le 3+\cos x \le 4$

$\Rightarrow \dfrac{2}{x^4} \le \dfrac{3+\cos x}{x^4} \le \dfrac{4}{x^4}$ & $\lim\limits_{x\to\infty} \dfrac{2}{x^4} = 0 = \lim\limits_{x\to\infty} \dfrac{4}{x^4}$

$\Rightarrow \lim\limits_{x\to\infty} \dfrac{3+\cos x}{x^4} = 0$ by squeeze principle.

3. Sketch the following graph. Make sure to CLEARLY state intercepts, asymptotes, along with the domain and range.

$$y = \frac{x+2}{x^2(x+1)}$$

x-int: $y=0$ when $x=-2$

y-int: $x=0$ DNE

V.A.: $\frac{2}{0^+} = \lim_{x\to 0^-} y = +\infty$, $\frac{2}{0^+} = \lim_{x\to 0^+} y = +\infty$ $\Big\}$ $x=0$

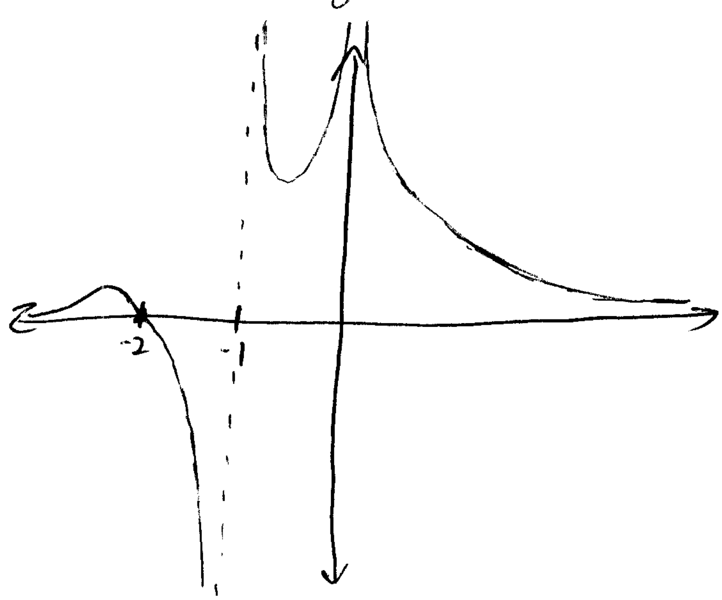
$\frac{3}{0^-} = \lim_{x\to -1^-} y = -\infty$, $\frac{3}{0^+} = \lim_{x\to -1^+} y = +\infty$ $\Big\}$ $x=-1$

H.A.: $\lim_{x\to\infty} \frac{x+2}{x^3+x^2} \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\cancel{\frac{1}{x^2}}^{0} + \cancel{\frac{2}{x^3}}^{0}}{1+\cancel{\frac{1}{x}}_{0}} = 0$

$\lim_{x\to-\infty} \frac{x+2}{x^3+x^2} \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to-\infty} \frac{\cancel{\frac{1}{x^2}}^{0} + \cancel{\frac{2}{x^3}}^{0}}{1+\cancel{\frac{1}{x}}_{0}} = 0$ $\Big\}$ $y=0$

Domain: $\forall x$ $x \neq 0, -1$

Range: $\forall y$



4. Use the Intermediate Value Theorem to show that the equation $x^3 = 10 + \sqrt{x}$ is solvable. This is a writing exercise.

$$x^3 = 10 + \sqrt{x} \iff x^3 - \sqrt{x} - 10 = 0$$

Let $f(x) = x^3 - \sqrt{x} - 10$ & $m = 0$, where f is continuous (poly + sqrt is cont.)

Since $f(0) = -10 < 0$ & $f(9) = 9^3 - 3 - 10 > 0$

$\Rightarrow f(0) < m = 0 < f(9)$. Choose interval $[0, 9]$

By IMVT, there exists @ least one c, $0 \le c \le 9$, s.t. $f(c) = 0 \iff c^3 = 10 + \sqrt{c}$. Thus, equation is solvable.

5.

(a) Consider the following function

$$f(x) = \begin{cases} \frac{x^2+x-2}{x^2-1} & \text{if } x < 1 \\ 3/2 & \text{if } x = 1 \\ 2x - 1/2 & \text{if } x > 1 \end{cases}$$

Use the three step definition of continuity to determine if f is continuous at $x = 1$.

1) $f(1) = 3/2$

2) From 2a $\lim_{x \to 1^-} f(x) = \frac{3}{2}$
Also, $\lim_{x \to 1^+} f(x) = 2 - \frac{1}{2} = \frac{3}{2}$
$\Rightarrow \lim_{x \to 1} f(x) = \frac{3}{2}$

3) $\lim_{x \to 1} f(x) = f(1)$

Hence, f is cont. @ $x = 1$

(b) Consider the following function

$$f(x) == \begin{cases} x^2 + K & \text{if } x < -1 \\ x + K^2 & \text{if } x > -1 \end{cases}$$

Determine all constants K such that the function is continuous for all values of x.

For continuity, $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

$\Leftrightarrow 1 + k = -1 + k^2 \Rightarrow k^2 + k - 2 = 0$

$\Rightarrow (k+2)(k-1) = 0 \Rightarrow \boxed{k = -2, 1}$

4.) Consider the following function $f(x) = \begin{cases} \dfrac{x^2-3x}{x^2-9}, & \text{if } x \neq 3, -3 \\ \frac{1}{2}, & \text{if } x = 3 \\ 0, & \text{if } x = -3 . \end{cases}$

Determine if f is continuous at $x = 3$.

i.) $f(3) = 1/2$

ii.) $\lim_{x \to 3} f(x) = \lim_{x\to 3} \frac{x^2-3x}{x^2-9} \overset{\text{"}0/0\text{"}}{=} \lim_{x \to 3} \frac{x(\cancel{x-3})}{(\cancel{x-3})(x+3)}$

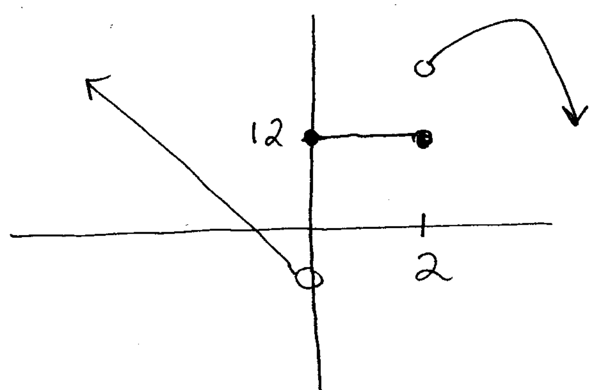
$= 3/6 = 1/2$

iii.) $\lim_{x\to 3} f(x) = f(3)$ so

f is continuous at $x = 3$.

5.) Using limits, determine the value(s) of constants A and B so that the following function is continuous for all values of x :

$f(x) = \begin{cases} Ax+B, & \text{if } x < 0 \\ 12, & \text{if } 0 \leq x \leq 2 \\ Bx^2 - A, & \text{if } x > 2 . \end{cases}$



$\underline{\lim_{x\to 0^-} (Ax+B) = 12}$ and $\underline{\lim_{x \to 2^+} (Bx^2 - A) = 12}$

$\to B = 12$ and $4B - A = 12 \to$

$48 - A = 12 \to A = 36$

6.

(a) State the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Use the definition of the derivative to differentiate the function

$$f(x) = \frac{x}{x+7}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{x+h+7} - \frac{x}{x+7}}{h} \cdot \frac{(x+7)(x+h+7)}{(x+7)(x+h+7)}$$

$$= \lim_{h \to 0} \frac{(x+h)(x+7) - x(x+h+7)}{h(x+7)(x+h+7)}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \cancel{7x} + \cancel{hx} + 7h - \cancel{x^2} - \cancel{hx} - \cancel{7x}}{h(x+7)(x+h+7)}$$

$$= \lim_{h \to 0} \frac{7\cancel{h}}{\cancel{h}(x+7)(x+h+7)} = \frac{7}{(x+7)^2}$$