2. Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

(a) $y = \pi + \dfrac{1}{(4x+1)^5}$

$$y' = 0 - 5(4x+1)^{-6} \cdot 4 \overset{\text{or}}{=} -\frac{20}{(4x+1)^6}$$

(b) $g(x) = x^5 8^{-x^2}$

$$g'(x) = 5x^4 8^{-x^2} + x^5 8^{-x^2} \ln 8 \cdot (-2x)$$

(c) $f(x) = \dfrac{x \sec x}{5x + \cot x}$

$$f'(x) = \frac{(\sec x + x \sec x \tan x)(5x + \cot x) - (x \sec x)(5 - \csc^2 x)}{(5x + \cot x)^2}$$

(d) $y = \log_4(\log_3(2x))$

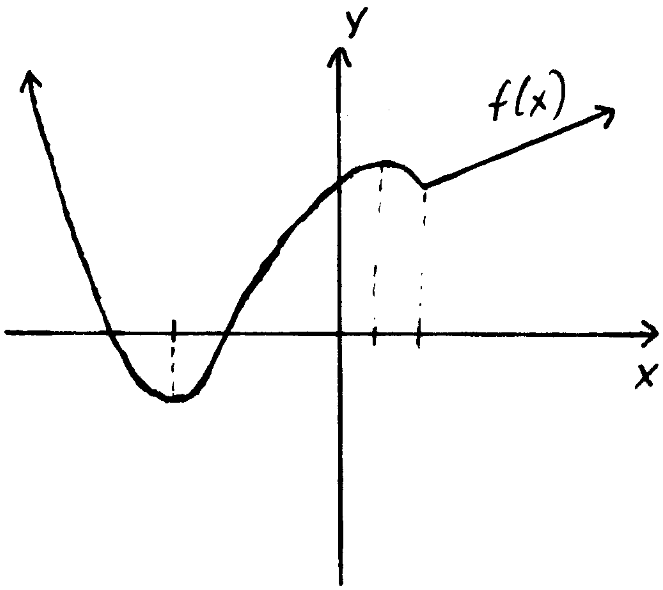
$$y' = \frac{1}{\log_3(2x)} \cdot \frac{1}{\ln 4} \cdot \frac{1}{2x} \cdot \frac{1}{\ln 3} \cdot 2$$

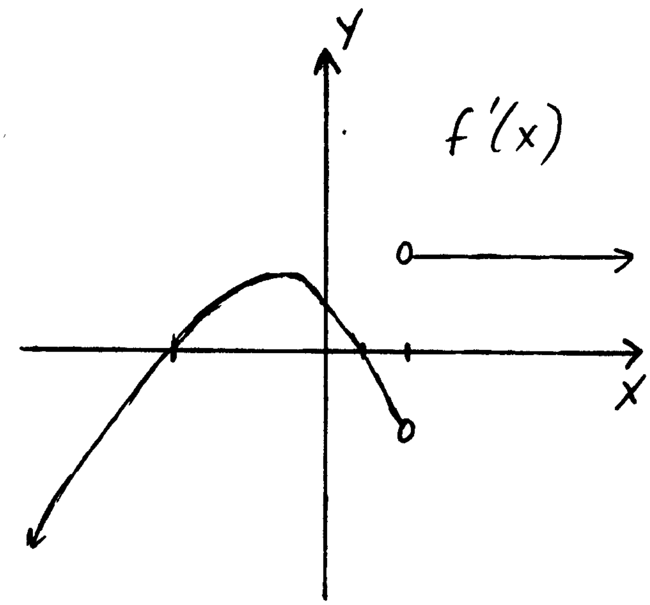
(e) $y = (2x)^{x+3}$

$$\ln y = (x+3) \ln 2x \Rightarrow \frac{y'}{y} = 1 \ln 2x + (x+3)\frac{1}{\cancel{2}x} \cdot \cancel{2}$$

$$\Rightarrow y' = y\left(\ln 2x + \frac{x+3}{x}\right) \Rightarrow y' = (2x)^{x+3}\left(\ln 2x + \frac{x+3}{x}\right)$$

3. Sketch the graph of f' using the graph of f.





4. Consider the following equation for a curve

$$x^2 - 2y^3 = xy$$

(a) Find the slope of the given curve at (2,1)

$$D_x[x^2-2y^3=xy] \Rightarrow 2x-6y^2y'=y+xy'$$

$$\Rightarrow 2x-y=6y^2y'+xy' \Rightarrow (6y^2+x)y'=2x-y$$

$$\Rightarrow y'=\frac{2x-y}{6y^2+x} \Rightarrow y'=\frac{3}{8} \text{ @ } (2,1)$$

(b) Find the equation of the tangent line at (2,1) supposing you obtained in part (a)

$$\frac{dy}{dx}=\frac{3}{2}$$

$$y-1=\frac{3}{2}(x-2) \Rightarrow y=\frac{3}{2}x-3+1$$

$$\Rightarrow y=\frac{3}{2}x-2$$

5.

(a) State one of the three limit definitions for the number e

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

or $e = \lim_{n \to -\infty} \left(1 + \frac{1}{n}\right)^n$ or $e = \lim_{z \to 0} (1 + z)^{1/z}$

(b) Evaluate the following limit by using one of the three limit definitions for the number e

$$\lim_{n \to \infty} \left(1 - \frac{3}{n}\right)^n$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{\frac{-n}{3}}\right)^{n \cdot \frac{-3}{-3}} = \lim_{n \to \infty} \left[\left(1 + \frac{1}{\frac{-n}{3}}\right)^{\frac{-n}{3}}\right]^{-3}$$

$$= \left[\lim_{n \to \infty} \left(1 + \frac{1}{\frac{-n}{3}}\right)^{\frac{-n}{3}}\right]^{-3} = e^{-3}$$

6.

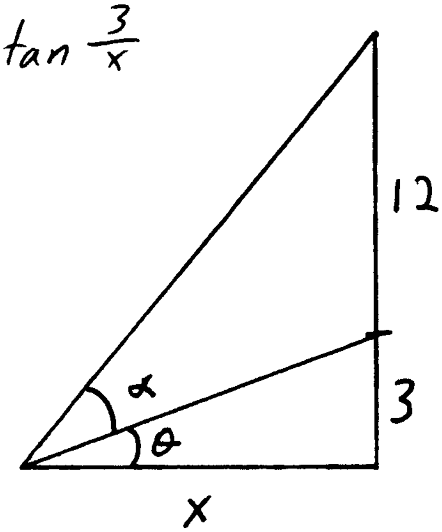
(a) Consider the given diagram. Write α as a function of x.

$\tan(\alpha+\theta) = \frac{15}{x}$ & $\tan\theta = \frac{3}{x}$

$\Rightarrow \alpha+\theta = \arctan\frac{15}{x}$ & $\theta = \arctan\frac{3}{x}$

$\Rightarrow \alpha = \arctan\frac{15}{x} - \theta$

$\Rightarrow \alpha = \arctan\frac{15}{x} - \arctan\frac{3}{x}$



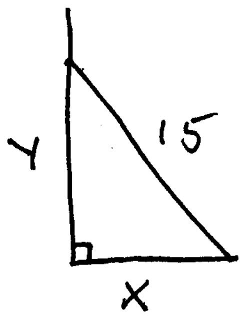
(b) Assuming $\alpha(t)$ and $x(t)$ are functions of time, use implicit differentiation with respect to time to find the related rates equation for the above diagram, supposing you obtained in part (a)

$$\alpha = \arctan\frac{x+3}{4} - \arctan\frac{3}{4}$$

$D_t\left[\alpha = \arctan\frac{x+3}{4} - \arctan\frac{3}{4}\right]$

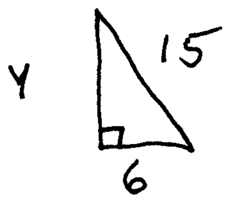
$$\frac{d\alpha}{dt} = \frac{1}{1+\left(\frac{x+3}{4}\right)^2}\left(\frac{1}{4}\right)\frac{dx}{dt}$$

8.) A 15-foot ladder is leaning against a wall. If the base of the ladder is pushed toward the wall at the rate of 2 ft./sec., at what rate is the top of the ladder moving up the wall when the base of the ladder is 6 ft. from the wall ?



assume $\frac{dX}{dt} = -2$ ft./sec.; find $\frac{dY}{dt}$ when $X = 6$ ft. :

$$X^2 + Y^2 = 15^2 \xrightarrow{D} 2X \cdot \frac{dX}{dt} + 2Y \frac{dY}{dt} = 0$$



$$6^2 + Y^2 = 15^2 \to Y = \sqrt{189} \approx 13.7$$

$$(6)(-2) + \sqrt{189} \cdot \frac{dY}{dt} = 0 \to$$

$$\frac{dY}{dt} = \frac{12}{\sqrt{189}} \approx 0.87 \text{ ft./sec.}$$

3.) You are standing on the top edge of a building which is 96 ft. high. You throw an apple straight UP at 80 ft./sec. and watch as it falls back to the ground.

a.) Assume that the acceleration due to gravity is $s''(t) = -32$ ft./sec.^{2} . Derive velocity, $s'(t)$, and height (above ground), $s(t)$, formulas for this apple.

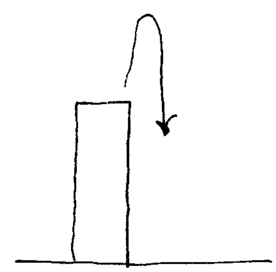
$$s''(t) = -32 \to$$

$$s'(t) = -32t + C \text{ (and } s'(0) = 80 \text{ ft./sec.)}$$

$$\to C = 80 \to \boxed{s'(t) = -32t + 80} ;$$

$$s(t) = -16t^2 + 80t + C \text{ (and } s(0) = 96 \text{ ft.)}$$

$$\to C = 96 \to \boxed{s(t) = -16t^2 + 80t + 96}$$



b.) In how many seconds will the apple strike the ground ?

strike ground : $s(t) = 0 \to$

$$-16t^2 + 80t + 96 = 0 \to -16(t^2 - 5t - 6) = 0 \to$$

$$-16(t-6)(t+1) = 0 \to t = 6 \text{ sec.}$$

c.) How high does the apple go ?

highest point : $s'(t) = 0 \to -32t + 80 = 0 \to$

$$\boxed{t = 2.5 \text{ sec.}} \to$$

$$s(2.5) = -16(2.5)^2 + 80(2.5) + 96 = 196 \text{ ft.}$$

8. Use differentials to estimate the value of $\sqrt[3]{26.5}$

Let $f(x)=\sqrt[3]{x}$

$x: \overset{x}{27} \to \overset{x+\Delta x}{26.5} \Rightarrow \Delta x = -0.5 = -\frac{1}{2}$

$f'(x) = \frac{1}{3x^{2/3}}$

$df = f'(27)\Delta x = \frac{1}{3(27)^{2/3}}\left(-\frac{1}{2}\right) = \frac{-1}{54}$

$\Rightarrow \sqrt[3]{26.5} = f(x+\Delta x) \approx f(x)+df = f(27)-\frac{1}{54} = 3-\frac{1}{54}$

$\Rightarrow \sqrt[3]{26.5} \approx 2\frac{53}{54}$

9. Use a differential to linearize the following function at the center point of $x = 5$

$$f(x) = \sqrt{x+4}$$

$f'(x) = \frac{1}{2\sqrt{x+4}}$

Let $x: \overset{x}{5} \to \overset{x+\Delta x}{5+h} \Rightarrow \Delta x = h$

$df = f'(5)\Delta x = \frac{1}{6}h$

$\Rightarrow \sqrt{(5+h)+4} = f(x+\Delta x) \approx f(x)+df = f(5)+\frac{1}{6}h = 3+\frac{1}{6}h$

$\Rightarrow \sqrt{h+9} \approx 3+\frac{1}{6}h$

The following extra credit problem is worth This problem is OPTIONAL and you are advised to finish the rest of the test before trying this problem.

1. State and prove the product rule for differentiation

Product Rule: $D[f(x)\cdot g(x)] = f'(x)\,g(x) + f(x)\cdot g'(x)$

Let $h(x) = f(x)\cdot g(x)$

$$h'(x) = \lim_{\Delta x\to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) \overbrace{- f(x)\,g(x+\Delta x) + f(x)\,g(x+\Delta x)}^{\text{Addition by } 0} - f(x)\,g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\,g(x+\Delta x) + f(x)\,\frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x)\,g(x) - f(x)\,g'(x)$$