Math 21A
Vogler
The Mean Value Theorem (MVT) and Other Important Theorems

*Definition*: Function $f$ takes on its *maximum* value at $x = c$ if $f(x) < f(c)$ for $x \neq c$. Function $f$ takes on its *minimum* value at $x = c$ if $f(c) < f(x)$ for $x \neq c$.

*Theorem A*: If function $f$ is differentiable at $x = c$, then $f$ is continuous at $x = c$.

*Theorem B*: Assume that function $f$ is differentiable and takes on its maximum value at $x = c$. Then $f'(c) = 0$.

*Theorem C*: Assume that function $f$ is differentiable and takes on its minimum value at $x = c$. Then $f'(c) = 0$. 
**Rolle's Theorem**: Assume that function $f$ is continuous on the closed interval $[a, b]$, differentiable on the open interval $(a, b)$, and $f(a) = f(b)$. Then there is at least one number $c$, $a < c < b$, so that $f'(c) = 0$.

**Proof**: Since $f$ is continuous on a closed interval $[a, b]$, $f$ has a maximum value $M$ and a minimum value $m$. This follows from the Maximum/Minimum Value Theorems discussed earlier in this course.

1. If $m = M$, then $f(x) = k$ for some constant $k$ and all values $x$ in $[a, b]$. Thus, $f'(x) = 0$ for all values of $x$ in $[a, b]$. It follows that $f'(c) = 0$ for some value of $c$, $a < c < b$.

2. If $m < M$, then both $m$ and $M$ cannot occur at endpoints $a$ and $b$ since $f(a) = f(b)$. Thus, at least one occurs in the interior of the interval at $x = c$. It follows from Theorems B and C that $f'(c) = 0$.

**Mean Value Theorem (MVT)**: Assume that function $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then there is at least one number $c$, $a < c < b$, so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ 

**Proof**: The equation of line $L$ in the diagram is

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a},$$

so that

$$y = \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a).$$

Define a new function

$$s(x) = f(x) - y = f(x) - \left\{ \frac{f(b) - f(a)}{b - a} \cdot (x - a) + f(a) \right\}.$$ 

This function is differentiable on the open interval $(a, b)$ since it is the difference of differentiable functions. This function is continuous on the closed interval $[a, b]$ since it is the difference of continuous functions. In addition, $s(a) = 0$ and $s(b) = 0$. It follows from Rolle's Theorem that there exists a number $c$, $a < c < b$, so that $s'(c) = 0$. Since

$$s'(x) = f'(x) - \left\{ \frac{f(b) - f(a)}{b - a} \cdot (1) + (0) \right\} = 0,$$

it follows that

$$s'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \rightarrow \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$ 

QED
**Theorem D**: Assume that \( f'(x) = 0 \) for all values of \( x \) in the closed interval \([a, b]\). Then \( f(x) = k \), a constant function on \([a, b]\).

**Proof**: Consider any two arbitrary \( x \)-values \( w \) and \( z \) in \([a, b]\) with \( w < z \). Consider the restriction of \( f \) to the new interval \([w, z]\). Since \( f \) is differentiable on the closed interval \([w, z]\) (and hence on the open interval \((w, z)\)), it follows from Theorem A that \( f \) is continuous on the open interval \((w, z)\). By the MVT there is at least one number \( c \), \( w < c < z \), so that

\[
f'(c) = \frac{f(z) - f(w)}{z - w} \quad \rightarrow \quad \frac{f(z) - f(w)}{z - w} = 0 \quad \text{(Since } f'(c) = 0 \text{)} \quad \rightarrow \quad f(z) - f(w) = 0 \quad \rightarrow \quad f(z) = f(w).
\]

Since \( w \) and \( z \) were chosen arbitrarily, it must be that \( f(x) = k \) for some constant \( k \) and for all values of \( x \) in the closed interval \([a, b]\). QED

---

**Theorem E**: Assume that \( f'(x) = g'(x) \) for all values of \( x \) in the closed interval \([a, b]\). Then \( f(x) = g(x) + c \) for some constant \( c \).

**Proof**: Since \( f'(x) = g'(x) \) \( \rightarrow \)

\[
f'(x) - g'(x) = 0 \quad \rightarrow \quad D(f(x) - g(x)) = 0 \quad \rightarrow \quad f(x) - g(x) = c \text{ for some constant } c \text{ (by Theorem D)} \quad \rightarrow \quad f(x) = g(x) + c \text{ for some constant } c.
\]

QED
Math 21A
Vogler
Mean Value Theorem-- An Application

Let $s(t)$ be the total miles you have driven after $t$ hours on Interstate 80 from Davis to San Francisco. Assume also that $s(1/3) = 10$ miles and $s(1.2) = 82$ miles. At a check-point along the way you admit these facts to a highway patrol officer, who, while a student at UC Davis, took Math 21A. The officer makes a few quick calculations, smiles, mumbles something about the Mean Value Theorem, then politely prepares to write you a traffic ticket. Explain why.