Centroid— The Balance Point \((\bar{x}, \bar{y})\) of a Flat Plate of Uniform (Constant) Density

Consider a flat region \(R\) whose height at \(x\), \(a \leq x \leq b\), is given to be \(h(x)\) and whose width at \(y\), \(c \leq y \leq d\), is given to be \(w(y)\). Assume the density at point \((x, y)\) is \(\delta(x, y) = k\), a constant. The standard formulas for the coordinates of the centroid \((\bar{x}, \bar{y})\) of region \(R\) are

\[
\bar{x} = \frac{\int_a^b x h(x) \, dx}{\int_a^b h(x) \, dx} \quad \text{and} \quad \bar{y} = \frac{\int_c^d y w(y) \, dy}{\int_c^d w(y) \, dy}
\]

Following are two sets of alternate formulas and the corresponding regions.

\[
\bar{x} = \frac{\int_a^b x(f(x) - g(x)) \, dx}{\int_a^b (f(x) - g(x)) \, dx} \quad \text{and} \quad \bar{y} = \frac{\int_a^b (1/2)((f(x))^2 - (g(x))^2) \, dx}{\int_a^b (f(x) - g(x)) \, dx}
\]
\[ \bar{x} = \frac{\int_c^d \left( \frac{1}{2} \right) (g(y)^2 - h(y)^2) \, dy}{\int_c^d (g(y) - h(y)) \, dy} \quad \text{and} \quad \bar{y} = \frac{\int_c^d y(g(y) - h(y)) \, dy}{\int_c^d (g(y) - h(y)) \, dy} \]
Math 21B
Vogler

Center of Mass—The Balance Point \((\bar{x}, \bar{y})\) of a Flat Plate of Variable Density

Consider a flat region \(R\) bounded above by the graph of \(y = f(x)\) and below by the graph of \(y = g(x)\) for \(a \leq x \leq b\). Assume the density at point \((x, y)\) is \(\delta(x, y) = k(x)\), a function of \(x\) only (not \(y\)). The **standard formulas** for the coordinates of the centroid \((\bar{x}, \bar{y})\) of region \(R\) are

\[
\bar{x} = \frac{\int_{a}^{b} x(f(x) - g(x))\delta(x, y)\,dx}{\int_{a}^{b} (f(x) - g(x))\delta(x, y)\,dx} \quad \text{and} \quad \bar{y} = \frac{\int_{a}^{b} (1/2)((f(x))^2 - (g(x))^2)\delta(x, y)\,dx}{\int_{a}^{b} (f(x) - g(x))\delta(x, y)\,dx}
\]

**REMARK**: The integral \(\int_{a}^{b} (f(x) - g(x))\delta(x, y)\,dx\) represents the TOTAL MASS of the plate with variable density.