**Mean Value Theorem for Integrals**: If $f$ is a continuous function on the closed interval $[a, b]$, then there is at least one number $c$, $a \leq c \leq b$, so that

$$f(c)(b-a) = \int_a^b f(x) \, dx$$

**Proof**: Since $f$ is a continuous function on the closed interval $[a, b]$, by the Maximum- and Minimum-Value Theorems, $f$ has a maximum value $M$ and a minimum value $m$ on $[a, b]$, i.e.,

$$m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a)$$

so that

$$m \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq M$$

call this number $y_o$.

By the Intermediate Value Theorem (p. 99) there is at least one number $c$, $a \leq c \leq b$, so that

$$f(c) = y_o, \text{ i.e., } f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

so that

$$f(c)(b-a) = \int_a^b f(x) \, dx$$

**First Fundamental Theorem of Calculus (FTC1)**: Assume that $f$ is a continuous function on the closed interval $[a, b]$ and that $F(x) = \int_a^x f(t) \, dt$. Then $F'(x) = f(x)$.

**Proof**: Consider $F(x) = \int_a^x f(t) \, dt$ as the area under the graph of $f$ above the interval $[a, x]$. Then $F(x+h)$ is the area under the graph of $f$ above the interval $[a, x+h]$ and $F(x+h) - F(x)$ is the area of the "thin strip" from $x$ to $x+h$, i.e., $F(x+h) - F(x) = \int_x^{x+h} f(t) \, dt$. By the Mean Value Theorem for integrals there is at least one number $c$, $x \leq c \leq x+h$, so that

$$f(c) \cdot h = \int_x^{x+h} f(t) \, dt$$
The derivative of $F(x)$ can now be computed as

\[
F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_{x}^{x+h} f(t) \, dt}{h} = \lim_{h \to 0} \frac{f(c) \, h}{h} = \lim_{h \to 0} f(c) \quad \text{(Recall that } x \leq c \leq x + h.\text{)}
\]

\[= f(x).\]

**Second Fundamental Theorem of Calculus (FTC2)**: Let $f$ be a continuous function on the closed interval $[a, b]$. Assume that $F(x)$ is an antiderivative of $f(x)$, i.e., assume that $F'(x) = f(x)$. Then

\[
\int_{a}^{b} f(x) \, dx = F(x) \bigg|_{a}^{b} = F(b) - F(a).
\]

**Proof**: Let $A(x) = \int_{a}^{x} f(t) \, dt$. Then $A(a) = 0$, $A(b) = \int_{a}^{b} f(t) \, dt$, and $A'(x) = f(x)$ by FTC1. But $F'(x) = f(x)$. By Corollary 2 (p. 333) to the Mean Value Theorem $F(x) = A(x) + C$ for any constant $C$, or

\[A(x) = F(x) - C.\]

Then

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, dt = A(b) = A(b) - A(a) = (F(b) - C) - (F(a) - C) = F(b) - F(a) = F(x) \bigg|_{a}^{b}.
\]