

Math 21B
Vogler
Recursion (Reduction) Formulas

Reduction (Recursion) formulas are used to write a difficult integral in simpler form in successive steps. The following example illustrates this idea.

EXAMPLE of RECURSION:

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

HOW TO USE IT:

$$\begin{aligned} \int \sin^4 x \, dx \quad (\text{Use } n = 4) &= -\frac{\sin^3 x \cdot \cos x}{4} + \frac{3}{4} \int \sin^2 x \, dx \quad (\text{Use } n = 2) \\ &= -\frac{\sin^3 x \cdot \cos x}{4} + \frac{3}{4} \left[-\frac{\sin x \cdot \cos x}{2} + \frac{1}{2} \int \sin^0 x \, dx \right] \\ &= -\frac{\sin^3 x \cdot \cos x}{4} - \frac{3}{8} \sin x \cdot \cos x + \frac{3}{8} \int 1 \, dx \\ &= -\frac{\sin^3 x \cdot \cos x}{4} - \frac{3}{8} \sin x \cdot \cos x + \frac{3}{8} x \end{aligned}$$

DERIVE THE RECURSION:

$$\begin{aligned} \int \sin^n x \, dx &= \int \sin^{n-2} x \cdot \sin^2 x \, dx \\ &= \int \sin^{n-2} x \cdot (1 - \cos^2 x) \, dx \\ &= \int \sin^{n-2} x \, dx - \int (\sin^{n-2} x \cdot \cos x) \cos x \, dx \end{aligned}$$

(Let $u = \cos x$ and $dv = \sin^{n-2} x \cdot \cos x \, dx$,

then $du = -\sin x \, dx$ and $v = \frac{1}{n-1} \sin^{n-1} x$)

$$\begin{aligned} &= \int \sin^{n-2} x \, dx - \left[\frac{\sin^{n-1} x \cdot \cos x}{n-1} - \int \frac{\sin x \cdot \sin^{n-1} x}{n-1} \, dx \right] \\ &= \int \sin^{n-2} x \, dx - \frac{\sin^{n-1} x \cdot \cos x}{n-1} - \frac{1}{n-1} \int \sin^n x \, dx \quad (\text{Now the TWIST !}) \end{aligned}$$

$$\longrightarrow \int \sin^n x \, dx = \int \sin^{n-2} x \, dx - \frac{\sin^{n-1} x \cdot \cos x}{n-1} - \frac{1}{n-1} \int \sin^n x \, dx$$

$$\longrightarrow \left(1 + \frac{1}{n-1}\right) \int \sin^n x \, dx = \int \sin^{n-2} x \, dx - \frac{\sin^{n-1} x \cdot \cos x}{n-1}$$

$$\longrightarrow \frac{n}{n-1} \int \sin^n x \, dx = \int \sin^{n-2} x \, dx - \frac{\sin^{n-1} x \cdot \cos x}{n-1}$$

$$\longrightarrow \int \sin^n x \, dx = -\frac{\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

EXAMPLE: (Optional Practice) Derive each of the following recursions.

$$1.) \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

$$2.) \int \cos^n x \, dx = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

EXAMPLE: (Optional Practice) Use recursions to do the following integrals.

$$1.) \int \tan^3 x \, dx$$

$$2.) \int \cos^6 x \, dx$$