Math 21B
Vogler
Estimating the Value of a Definite Integral

Suppose that the integral $\int_a^b f(x)\,dx$ is too difficult (or impossible) to compute, or that you are simply required to estimate its exact value. The following two methods offer possible ways to compute an estimate and measure its accuracy.

1.) TRAPEZOIDAL RULE

a.) Divide the interval $[a,b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval.

c.) The Trapezoidal Estimate for $\int_a^b f(x)\,dx$ is

$$T_n = \frac{h}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)]\ .$$

d.) The Absolute Error is

$$|E_n| = \left|\int_a^b f(x)\,dx - T_n\right| \le (b-a)\frac{h^2}{12}\Big\{\max_{a\le x\le b}|f''(x)|\Big\} = \frac{(b-a)^3}{12n^2}\Big\{\max_{a\le x\le b}|f''(x)|\Big\}$$

2.) SIMPSON'S RULE (NOTE: For this method n MUST be an even integer !)

a.) Divide the interval $[a,b]$ into n equal parts, each of length $h = \dfrac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval.

c.) The Simpson Estimate for $\int_a^b f(x)\,dx$ is

$$S_n = \frac{h}{3}\,[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+...+4f(x_{n-3})+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$
.

d.) The Absolute Error is

$$|E_n| = \left|\int_a^b f(x)\,dx - S_n\right| \le (b-a)\frac{h^4}{180}\Big\{\max_{a\le x\le b}|f^{(4)}(x)|\Big\} = \frac{(b-a)^5}{180n^4}\Big\{\max_{a\le x\le b}|f^{(4)}(x)|\Big\}$$

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Simpson's Rule

Ex 1) Use S_4, Simpson's Rule with $n=4$, to estimate the exact value of $\int_{-5}^{-4} \frac{x+1}{x+3}\,dx$.

$f(x) = \frac{x+1}{x+3}$, $n=4$, $\Delta x = \frac{1}{4}$ -5, $-\frac{19}{4}$, $-\frac{9}{2}$, $-\frac{17}{4}$, -4

$$S_4 = \frac{\frac{1}{4}}{3}\left[f(-5) + 4f\left(\tfrac{-19}{4}\right) + 2f\left(\tfrac{-9}{2}\right) + 4f\left(\tfrac{-17}{4}\right) + f(-4)\right]$$

$$= \frac{1}{12}\left[2 + 4\left(\tfrac{15}{7}\right) + 2\left(\tfrac{7}{3}\right) + 4\left(\tfrac{13}{5}\right) + 3\right] \simeq 2.3865$$

Exact Value: $\int_{-5}^{-4} \frac{x+1}{x+3}\,dx = 1 + \ln 4 \simeq 2.3863$

Absolute Error $|E_4| = \left|\int_{-5}^{-4} \frac{x+1}{x+3}\,dx - S_4\right| = 0.0002$

2) What should n be in order that S_n, Simpson's Rule with n subdivisions, estimate the exact value of $\int_{-5}^{-4} \frac{x+1}{x+3}\,dx$ with absolute error of at most 0.00001?

Absolute Error $|E_n| \le \frac{(b-a)^5}{180 n^4} \cdot \max_{a \le x \le b} |f^{(4)}(x)|$

$f(x) = \frac{x+1}{x+3}$, $f'(x) = 2(x+3)^{-2}$, $f''(x) = -4(x+3)^{-3}$, $f'''(x) = 12(x+3)^{-4}$,

$$f^{(4)}(x) = \frac{-48}{(x+3)^5} \Rightarrow \max_{-5 \le x \le -4} |f^{(4)}(x)| = \frac{48}{|(-4)+3|^5} = 48$$

$$|E_n| \le \frac{(-4-(-5))^5}{180 n^4} \cdot 48 = \frac{12}{45 n^4} \le 0.00001$$

$$\Rightarrow n^4 \ge \frac{12}{45(0.00001)} \Rightarrow n \ge \left[\frac{12}{45(0.00001)}\right]^{1/4} \simeq 12.7$$

Hence, $\boxed{\text{use } n = 14}$