Section 8.3

1.) $\displaystyle\int \frac{dX}{\sqrt{3^2+X^2}}$ $\quad$ (Let $X = 3\tan\theta \rightarrow$ $dX = 3\sec^2\theta\, d\theta$)

$$= \int \frac{3\sec^2\theta\, d\theta}{\sqrt{3^2+3^2\tan^2\theta}} = \int \frac{\cancel{3}\sec^2\theta}{\cancel{3}\sqrt{1+\tan^2\theta}}\, d\theta$$

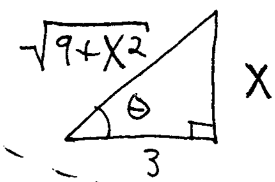
$$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\, d\theta = \int \frac{\sec^2\theta}{\sec\theta}\, d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{\sqrt{9+X^2}}{3} + \frac{X}{3}\right| + c$$

$\tan\theta = X/3 \rightarrow$

$\theta = \arctan(X/3)$



4.) $\displaystyle\int_0^2 \frac{dx}{8+2x^2}$

$$= \frac{1}{2}\int_0^2 \frac{1}{2^2+x^2}\, dx = \frac{1}{2}\cdot\frac{1}{2}\arctan\frac{x}{2}\Big|_0^2$$

$$= \frac{1}{4}\arctan 1 - \frac{1}{4}\arctan 0$$

$$= \frac{1}{4}\left(\frac{\pi}{4}\right) - \frac{1}{4}(0) = \frac{\pi}{16}$$

5.) $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{3^2-x^2}} = \arcsin\frac{x}{3}\Big|_0^{3/2}$

$$= \arcsin\frac{1}{2} - \cancel{\arcsin 0}^{\,0} = \frac{\pi}{6}$$

8.) $\displaystyle\int \sqrt{1-9t^2}\, dt = \int \sqrt{1-(3t)^2}\, dt$

(Let $3t = \sin\theta \rightarrow 3\,dt = \cos\theta\, d\theta \rightarrow$
$dt = \frac{1}{3}\cos\theta\, d\theta$)

$$= \frac{1}{3}\int \sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= \frac{1}{3}\int \sqrt{\cos^2\theta}\cdot \cos\theta \, d\theta = \frac{1}{3}\int \cos\theta \cdot \cos\theta \, d\theta$$

$$= \frac{1}{3}\int \cos^2\theta \, d\theta = \frac{1}{3}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

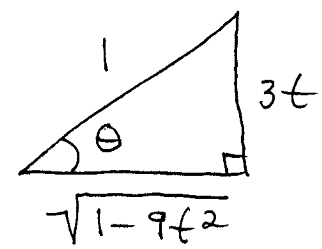
$$= \frac{1}{6}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{6}\left(\theta + \cancel{\tfrac{1}{2}}\cdot \cancel{2}\sin\theta\cos\theta\right) + C$$

$$= \frac{1}{6}\left(\arcsin 3t + 3t\cdot\sqrt{1-9t^2}\right) + C$$

$\sin\theta = 3t \rightarrow$

$\theta = \arcsin 3t$



11.) $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\, dy$

$$= \int \frac{\sqrt{y^2-7^2}}{y}\, dy$$

(Let $y = 7\sec\theta \rightarrow$
$dy = 7\sec\theta\tan\theta\, d\theta$)

$$= \int \frac{\sqrt{7^2\sec^2\theta - 7^2}}{\cancel{7}\cancel{\sec\theta}}\cdot \cancel{7}\cancel{\sec}\theta\tan\theta\, d\theta$$

$$= \int 7\sqrt{\sec^2\theta - 1}\cdot\tan\theta\, d\theta$$

$$= 7\int \sqrt{\tan^2\theta}\cdot\tan\theta\, d\theta$$

$$= 7\int \tan\theta\cdot\tan\theta\, d\theta$$

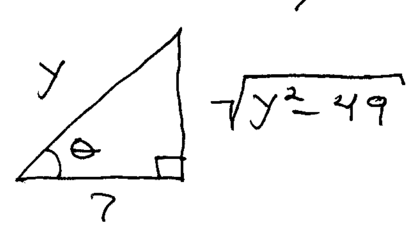
$$= 7\int \tan^2\theta\, d\theta = 7\int(\sec^2\theta - 1)\, d\theta$$

$$= 7(\tan\theta - \theta) + C$$

$$= 7\left(\frac{\sqrt{y^2-49}}{7} - \operatorname{arcsec}\frac{y}{7}\right) + C$$

$\sec\theta = y/7 \rightarrow$

$\theta = \operatorname{arcsec}\frac{y}{7}$



18.) $\int \frac{\sqrt{3^2-w^2}}{w^2}\,dw$ (Let $w = 3\sin\theta \to$ $dw = 3\cos\theta\,d\theta$)

$$= \int \frac{\sqrt{3^2 - 3^2\sin^2\theta}}{3^2\sin^2\theta} \cdot 3\cos\theta\,d\theta$$

$$= \int \frac{\cancel{3^2}\cdot\sqrt{1-\sin^2\theta}}{\cancel{3^2}\cdot\sin^2\theta}\cdot\cos\theta\,d\theta$$

$$= \int \frac{\sqrt{\cos^2\theta}\cdot\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta\cdot\cos\theta}{\sin^2\theta}\,d\theta$$

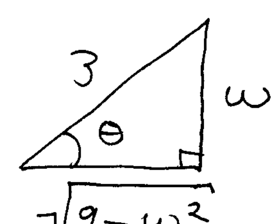
$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta$$

$$= \int(\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{9-w^2}}{w} - \arcsin\frac{w}{3} + C$$

$\sin\theta = \frac{w}{3} \to$

$\theta = \arcsin\frac{w}{3}$



24.) $\int \frac{\sqrt{1-x^2}}{x^4}\,dx$

(Let $x = \sin\theta \to dx = \cos\theta\,d\theta$)

$$= \int \frac{\sqrt{1-\sin^2\theta}\cdot\cos\theta}{\sin^4\theta}\,d\theta$$

$$= \int \frac{\cos\theta\cdot\cos\theta}{\sin^4\theta}\,d\theta = \int \frac{\cos^2\theta}{\sin^4\theta}\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\sin^2\theta}\,d\theta = \int \cot^2\theta\cdot\csc^2\theta\,d\theta$$

(Let $u = \cot\theta \to du = -\csc^2\theta\,d\theta \to$

13.) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}$ $\quad$ (Let $x = \sec\theta \rightarrow$
$dx = \sec\theta\tan\theta\, d\theta$)

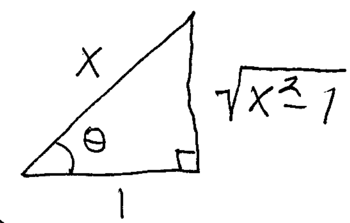
$$= \int \frac{\sec\theta\tan\theta \quad d\theta}{\sec^2\theta\,\sqrt{\sec^2\theta - 1}}$$

$$= \int \frac{\tan\theta \quad d\theta}{\sec\theta\,\sqrt{\tan^2\theta}} = \int \frac{\cancel{\tan\theta}}{\sec\theta\,\cancel{\tan\theta}}\, d\theta$$

$$= \int \cos\theta \; d\theta = \sin\theta + c$$

$$= \frac{\sqrt{x^2-1}}{x} + c$$

$\sec\theta = x \rightarrow$
$\theta = \operatorname{arcsec} x$



15.) $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

$= \displaystyle\int \frac{x^3}{\sqrt{x^2+2^2}}\,dx$ $\quad$ (Let $x = 2\tan\theta \rightarrow$
$dx = 2\sec^2\theta \;\, d\theta$)

$$= \int \frac{8\tan^3\theta \cdot 2\sec^2\theta}{\sqrt{2^2\cdot\tan^2\theta + 2^2}}\, d\theta$$

$$= \int \frac{16\tan^3\theta\,\sec^2\theta}{2\sqrt{\tan^2\theta + 1}}\, d\theta$$

$$= 8\int \frac{\tan^3\theta\,\sec^2\theta}{\sqrt{\sec^2\theta}}\, d\theta = 8\int \frac{\tan^3\theta\,\sec^2\theta}{\sec\theta}\, d\theta$$

$$= 8 \cdot \int \tan^3\theta\,\sec\theta \;\, d\theta$$

$$= 8\int \tan^2\theta \cdot \sec\theta\tan\theta \;\, d\theta$$

$$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta \; d\theta$$

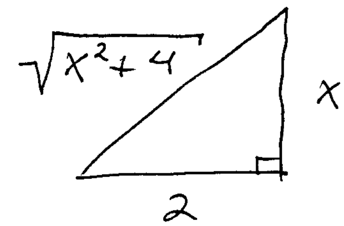
$$(\text{Let } u = \sec\theta \rightarrow du = \sec\theta\tan\theta\, d\theta)$$

$$= 8\int (u^2 - 1)\, du = 8\left(\frac{u^3}{3} - u\right) + c$$

$$= 8\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + c$$

$\tan\theta = x/2 \rightarrow$

$\theta = \arctan x/2$



$$= \frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\cdot\frac{\sqrt{x^2+4}}{2} + c$$

$$= \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + c$$

16.) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$ $\quad (\text{Let } x = \tan\theta \rightarrow dx = \sec^2\theta\, d\theta)$

$$= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sqrt{\sec^2\theta}}$$

$$= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\cdot\sec\theta} = \int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \int \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta}\, d\theta = \int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$$(\text{Let } u = \sin\theta \rightarrow du = \cos\theta\, d\theta)$$

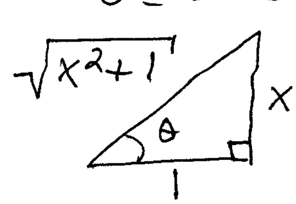
$$= \int \frac{1}{u^2}\, du = \frac{-1}{u} + c = \frac{-1}{\sin\theta} + c$$

$$= -\csc\theta + c$$

$\tan\theta = x \rightarrow$

$\theta = \arctan x$

$$= -\frac{\sqrt{x^2+1}}{x} + c$$



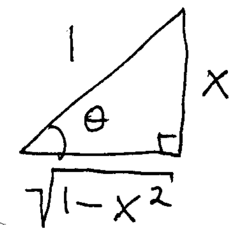
$$-du = \csc^2\theta \, d\theta \,)$$

$$= -\int u^2 \, du = \frac{-1}{3}u^3 + C$$

$$= \frac{-1}{3}\cot^3\theta + C$$

$\sin\theta = x \rightarrow$
$\theta = \arcsin x$

$$= \frac{-1}{3}\cdot\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$$



29.) $\displaystyle\int_0^{\ln 4} \frac{e^t \, dt}{\sqrt{e^{2t}+9}}$ (Let $u = e^t \rightarrow du = e^t dt$, $t: 0 \rightarrow \ln 4$ so $u: 1 \rightarrow 4$)

$$= \int_1^4 \frac{1}{\sqrt{u^2+3^2}} \, du$$ (Let $u = 3\tan\theta \longrightarrow$ $du = 3\sec^2\theta \, d\theta$)

$$= \int_{u=1}^{u=4} \frac{3\sec^2\theta \, d\theta}{\sqrt{3^2\tan^2\theta + 3^2}}$$

$$= \int_{u=1}^{u=4} \frac{\cancel{3}\sec^2\theta \, d\theta}{\cancel{3}\sqrt{\tan^2\theta + 1}} = \int_{u=1}^{u=4} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} \, d\theta$$

$$= \int_{u=1}^{u=4} \frac{\sec^2\theta}{\sec\theta} \, d\theta = \int_{u=1}^{u=4} \sec\theta \, d\theta$$

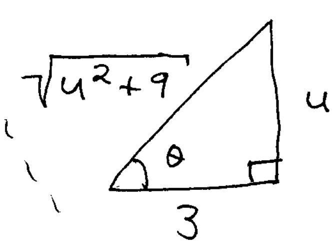
$$= \ln|\sec\theta + \tan\theta| \Big|_{u=1}^{u=4}$$

$\tan\theta = u/3$
$\rightarrow \theta = \arctan \frac{u}{3}$

$$= \ln\left|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}\right| \Big|_{u=1}^{u=4}$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right)$$

$$= \ln 3 - \ln\left(\frac{\sqrt{10}+1}{3}\right)$$



$$= \ln 3 - \left(\ln(\sqrt{10}+1) - \ln 3\right)$$
$$= 2\ln 3 - \ln(\sqrt{10}+1)$$
$$= \ln 3^2 - \ln(\sqrt{10}+1)$$
$$= \ln 9 - \ln(\sqrt{10}+1)$$

45.) $\displaystyle\int \sqrt{\frac{4-x}{x}}\,dx$ $\left(\text{Let } x = u^2 \xrightarrow{D} dx = 2u\,du\right)$

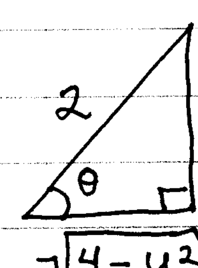
$$= \int \frac{\sqrt{4-u^2}}{\sqrt{u^2}}\cdot 2u\,du = 2\int \frac{\cancel{u}}{\cancel{u}}\sqrt{2^2-u^2}\,du$$

$\left(\text{Let } u = 2\sin\theta \xrightarrow{D} du = 2\cos\theta\,d\theta\right)$

$$= 2\int \sqrt{2^2 - 2^2\sin^2\theta}\cdot 2\cos\theta\,d\theta$$
$$= 4\int \sqrt{4(1-\sin^2\theta)}\,\cos\theta\,d\theta$$
$$= 8\int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta = 8\int \cos\theta\cdot\cos\theta\,d\theta$$
$$= 8\int \cos^2\theta\,d\theta = 8\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= 4\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$
$$= 4\theta + 2\cdot(2\sin\theta\cos\theta) + C$$
$$= 4\arcsin\left(\frac{u}{2}\right) + \cancel{4}\cdot\left(\frac{u}{\cancel{2}}\right)\cdot\frac{1}{2}\sqrt{4-u^2} + C$$
$$= 4\arcsin\left(\frac{1}{2}\sqrt{x}\right) + \sqrt{x}\sqrt{4-x} + C$$

$\sin\theta = \frac{u}{2}$

$\theta = \arcsin\left(\frac{u}{2}\right)$



47.) $\int \sqrt{x}\cdot\sqrt{1-x}\,dx$ (Let $x=u^2 \xrightarrow{D} dx=2u\,du$)

$= \int \sqrt{u^2}\cdot\sqrt{1-u^2}\cdot 2u\,du = 2\int u^2\sqrt{1-u^2}\,du$

(Let $u=\sin\theta \xrightarrow{D} du=\cos\theta\,d\theta$)

$= 2\int \sin^2\theta\cdot\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$

$= 2\int \sin^2\theta\cdot\sqrt{\cos^2\theta}\cdot\cos\theta\,d\theta$

$= 2\int \sin^2\theta\cos^2\theta\,d\theta$

$= 2\int (\sin\theta\cos\theta)^2\,d\theta$

$= 2\int \left(\frac{1}{2}\sin 2\theta\right)^2 d\theta = \frac{1}{2}\int \sin^2 2\theta\,d\theta$

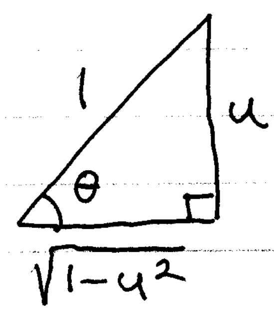
$= \frac{1}{2}\int \frac{1}{2}(1-\cos 4\theta)\,d\theta$

$\sin\theta = \frac{u}{1}$

$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin 4\theta\right) + C$

$\theta = \arcsin u$

$= \frac{1}{4}\theta - \frac{1}{16}\sin 2(2\theta) + C$



$= \frac{1}{4}\theta - \frac{1}{16}2\sin 2\theta\cos 2\theta + C$

$= \frac{1}{4}\theta - \frac{1}{8}(2\sin\theta\cos\theta)\cdot(2\cos^2\theta - 1) + C$

$= \frac{1}{4}\arcsin u - \frac{1}{4}u\sqrt{1-u^2}\cdot\left(2\left(\sqrt{1-u^2}\right)^2 - 1\right) + C$

$= \frac{1}{4}\arcsin\sqrt{x} - \frac{1}{4}\sqrt{x}\cdot\sqrt{1-x}\cdot(1-2x) + C$

52.) $(x^2+1)^2 \dfrac{dy}{dx} = \sqrt{x^2+1}$, $y(0)=1$ $\longrightarrow$

$$\int dy = \int \frac{(x^2+1)^{1/2}}{(x^2+1)^2}\, dx \longrightarrow$$

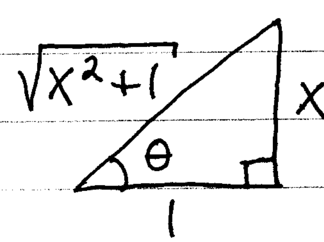
$$y = \int \frac{1}{(x^2+1)^{3/2}}\, dx \quad \left(\text{Let } x = \tan\theta \xrightarrow{D} dx = \sec^2\theta\, d\theta\right)$$

$$= \int \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^{3/2}} = \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\, d\theta$$

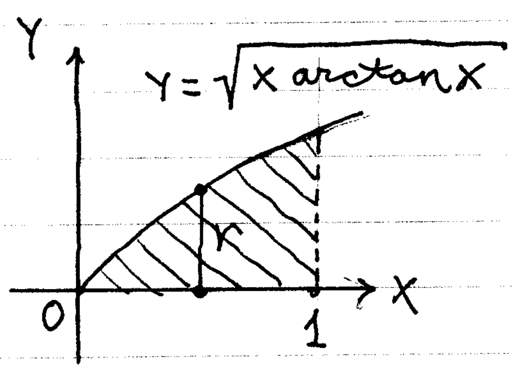
$$= \int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta = \int \frac{1}{\sec\theta}\, d\theta = \int \cos\theta\, d\theta$$

$$= \sin\theta + c \qquad \tan\theta = \frac{x}{1}$$

$$= \frac{x}{\sqrt{x^2+1}} + c$$



56.)



$$\text{Vol} = \pi \int_0^1 (\text{radius})^2\, dx$$

$$= \pi \int_0^1 \left(\sqrt{x \arctan x}\right)^2 dx$$

$$= \pi \int_0^1 x \arctan x\, dx$$

(Let $u = \arctan x$, $dv = x\,dx$

$\rightarrow$ $du = \dfrac{1}{1+x^2}\, dx$, $v = \frac{1}{2}x^2$)

$$= \frac{1}{2}x\cdot \arctan x \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx$$

$$\Big\{ \text{For } \int \frac{x^2}{1+x^2}\,dx \quad \left(\text{Let } x = \tan\theta \xrightarrow{D} dx = \sec^2\theta\, d\theta\right)$$

$$= \int \frac{\tan^2\theta}{1+\tan^2\theta}\cdot \sec^2\theta\, d\theta$$

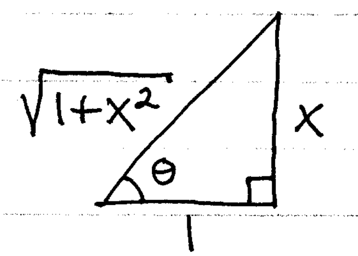
$$= \int \frac{\tan^2\theta}{\cancel{\sec^2\theta}}\cdot \cancel{\sec^2\theta}\, d\theta = \int \tan^2\theta\, d\theta$$

$$= \int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta + C$$

$$= x - \arctan x + C$$

$$\tan\theta = \frac{x}{1}$$

$$\theta = \arctan x$$



$$\Big\}$$

$$= \frac{1}{2}(1)\overset{\pi/4}{\arctan 1} - \frac{1}{2}(0)\overset{0}{\arctan 0}$$

$$- \frac{1}{2}\left[x - \arctan x\right]\Big|_0^1$$

$$= \frac{1}{8}\pi - \frac{1}{2}\left[(1 - \overset{\pi/4}{\arctan 1}) - (0 - \overset{0}{\arctan 0})\right]$$

$$= \frac{1}{8}\pi - \frac{1}{2} + \frac{1}{8}\pi = \frac{\pi}{4} - \frac{1}{2}$$

57.) a.) $\int x^3\sqrt{1-x^2}\,dx = \int x^2\cdot x\sqrt{1-x^2}\,dx$

(Let $u = x^2$, $dv = x\sqrt{1-x^2}\,dx$

$\rightarrow du = 2x\,dx$, $v = \frac{-1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}(1-x^2)^{3/2} = \frac{-1}{3}(1-x^2)^{3/2}$)

$$= \frac{-1}{3}x(1-x^2)^{3/2} - \frac{-2}{3}\int x(1-x^2)^{3/2}\,dx$$

$$= \frac{-1}{3}x(1-x^2)^{3/2} + \frac{2}{3}\cdot\frac{-1}{\cancel{2}}\cdot\frac{\cancel{2}}{5}(1-x^2)^{5/2} + C$$

$$= \frac{-1}{3}x(1-x^2)^{3/2} - \frac{2}{15}(1-x^2)^{5/2} + C$$

b.) $\int x^3\sqrt{1-x^2}\,dx = \int x^2 \cdot x\sqrt{1-x^2}\,dx$

(Let $u = 1-x^2 \xrightarrow{D} du = -2x\,dx \rightarrow -\frac{1}{2}du = x\,dx$ and $x^2 = 1-u$)

$$= \frac{-1}{2}\int (1-u)u^{1/2}\,du = \frac{-1}{2}\int (u^{1/2} - u^{3/2})\,du$$

$$= \frac{-1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C$$

$$= \frac{-1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C$$

c.) $\int x^3\sqrt{1-x^2}\,dx$ (Let $x = \sin\theta \xrightarrow{D} dx = \cos\theta\,d\theta$)

$$= \int \sin^3\theta\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$$

$$= \int \sin^3\theta\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= \int \sin^3\theta\,\cos^2\theta\,d\theta$$

$$= \int \sin\theta\cdot\sin^2\theta\cdot\cos^2\theta\,d\theta$$

$$= \int \sin\theta\,(1-\cos^2\theta)\cos^2\theta\,d\theta$$

(Let $u = \cos\theta \xrightarrow{D} du = -\sin\theta\,d\theta \rightarrow -du = \sin\theta\,d\theta$)

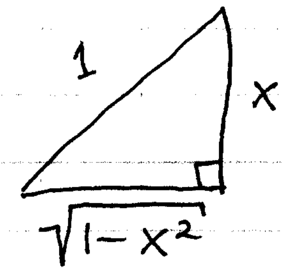
$$= -\int (1-u^2)u^2\,du = -\int (u^2 - u^4)\,du$$

$= -\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + c$

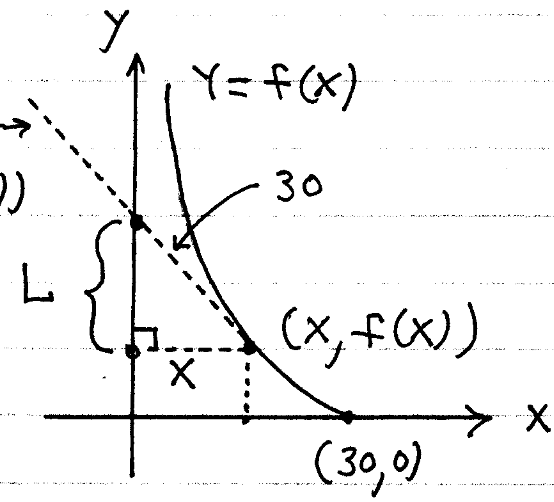
$= -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + c$

$= \frac{-1}{3}\left(\sqrt{1-x^2}\right)^3 + \frac{1}{5}\left(\sqrt{1-x^2}\right)^5 + c$

$\sin\theta = \frac{x}{1}$



58.) a.) tangent line at $(x, f(x))$ $\longrightarrow$



$x^2 + L^2 = 30^2 \longrightarrow$

$L = \sqrt{900 - x^2}$;

SLOPE of tangent line is

$$f'(x) = \frac{-L}{x} = \frac{-\sqrt{900-x^2}}{x}.$$

b.) $f'(x) = \dfrac{-\sqrt{900-x^2}}{x} \longrightarrow$

$$f(x) = \int \frac{-\sqrt{900-x^2}}{x}\,dx$$

(Let $x = 30\sin\theta \rightarrow dx = 30\cos\theta\, d\theta$)

$$= -\cancel{30}\int \frac{\sqrt{900 - 900\sin^2\theta}}{\cancel{30}\sin\theta}\cos\theta\, d\theta$$

$$= -\int \frac{\sqrt{900(1-\sin^2\theta)}}{\sin\theta}\cos\theta\, d\theta$$

$$= -30\int \frac{\sqrt{\cos^2\theta}}{\sin\theta}\cos\theta\, d\theta$$

$$= 30\int \frac{\cos^2\theta}{\sin\theta}\,d\theta = 30\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$= 30\int \left[\frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}\right] d\theta$$

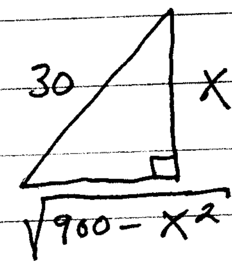
$$= 30\int (\csc\theta - \sin\theta)\,d\theta$$

$$= 30\left[\ln|\csc\theta - \cot\theta| + \cos\theta\right] + C$$

$$\sin\theta = \frac{x}{30}$$

$$= 30\ln\left|\frac{30}{x} - \frac{\sqrt{900-x^2}}{x}\right|$$

$$+ \cancel{30}\cdot\frac{1}{\cancel{30}}\sqrt{900-x^2} + C$$



$$= 30\ln\left|\frac{30-\sqrt{900-x^2}}{x}\right|$$

$$+ \sqrt{900-x^2} + C$$