Section 8.4

9.) $\int \frac{1}{1-x^2}\,dx = \int \frac{1}{(1-x)(1+x)}\,dx$

$= \int \left[\frac{A}{1-x} + \frac{B}{1+x}\right] dx$

$(A(1+x) + B(1-x) = 1$

Let $x=1$: $2A = 1 \rightarrow A = 1/2$

Let $x=-1$: $2B = 1 \rightarrow B = 1/2)$

$= \int \left[\frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\right] dx$

$= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + c$

12.) $\int \frac{2x+1}{x^2-7x+12}\,dx = \int \frac{2x+1}{(x-4)(x-3)}\,dx$

$= \int \left[\frac{A}{x-4} + \frac{B}{x-3}\right] dx$

$(A(x-3) + B(x-4) = 2x+1$

Let $x=3$: $-B = 7 \rightarrow B = -7$

Let $x=4$: $A = 9)$

$= \int \left[\frac{9}{x-4} + \frac{-7}{x-3}\right] dx$

$= 9\ln|x-4| - 7\ln|x-3| + c$

13.) $\int_4^8 \frac{y}{y^2-2y-3}\,dy = \int_4^8 \frac{y}{(y-3)(y+1)}\,dy$

$= \int_4^8 \left[\frac{A}{y-3} + \frac{B}{y+1}\right] dy$

$(A(y+1) + B(y-3) = y$

Let $y=-1$: $-4B = -1 \rightarrow B = 1/4$

Let $y=3$: $4A = 3 \rightarrow A = 3/4)$

$$= \int_4^8 \left[\frac{3/4}{y-3} + \frac{1/4}{y+1} \right] dy$$

$$= \left(\frac{3}{4} \ln|y-3| + \frac{1}{4} \ln|y+1| \right) \Big|_4^8$$

$$= \left(\frac{3}{4} \ln 5 + \frac{1}{4} \ln 9 \right) - \left(\frac{3}{4} \cancel{\ln 1}^{\,0} + \frac{1}{4} \ln 5 \right)$$

$$= \frac{1}{2} \ln 5 + \frac{1}{4} \ln 9 = \ln 5^{1/2} + \ln (3^2)^{1/4}$$

$$= \ln 5^{1/2} + \ln 3^{1/2} = \ln (5^{1/2} \cdot 3^{1/2})$$

$$= \ln 15^{1/2} = \frac{1}{2} (\ln 15)$$

16.) $\displaystyle\int \frac{x+3}{2x^3 - 8x} \, dx = \int \frac{x+3}{2x(x-2)(x+2)} \, dx$

$$= \frac{1}{2} \int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} \right] dx$$

$\Big(A(x-2)(x+2) + Bx(x+2) + Cx(x-2) = x+3$

Let $x=2$: $8B = 5 \rightarrow B = 5/8$

Let $x=-2$: $8C = 1 \rightarrow C = 1/8$

Let $x=0$: $-4A = 3 \rightarrow A = -3/4 \Big)$

$$= \frac{1}{2} \int \left[\frac{-3/4}{x} + \frac{5/8}{x-2} + \frac{1/8}{x+2} \right] dx$$

$$= \frac{1}{2} \left[-\frac{3}{4} \ln|x| + \frac{5}{8} \ln|x-2| + \frac{1}{8} \ln|x+2| \right] + C$$

19.) $\displaystyle\int \frac{1}{(x^2-1)^2} \, dx = \int \frac{1}{((x-1)(x+1))^2} \, dx$

$$= \int \frac{1}{(x-1)^2 (x+1)^2} \, dx$$

$$= \int \left[\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} \right] dx$$

$$\Big(A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2 = 1$$

Let $x=1$: $4B = 1 \rightarrow B = 1/4$

Let $x=-1$: $4D = 1 \rightarrow D = 1/4$

Let $x=0$: $-A + \frac{1}{4} + C + \frac{1}{4} = 1 \rightarrow$

$$\underline{-A + C = 1/2}$$

Let $x=2$: $9A + \frac{9}{4} + 3C + \frac{1}{4} = 1 \rightarrow$

$$\underline{9A + 3C = -3/2}$$

$12C = 3 \rightarrow C = 1/4$ and, ~~then~~ $A = -1/4$ $\Big)$

$$= \int \left[\frac{-1/4}{x-1} + \frac{1/4}{(x-1)^2} + \frac{1/4}{x+1} + \frac{1/4}{(x+1)^2} \right] dx$$

$$= -\frac{1}{4} \ln|x-1| - \frac{1}{4} \cdot \frac{1}{x-1} + \frac{1}{4} \ln|x+1| - \frac{1}{4} \cdot \frac{1}{x+1} + C$$

20.) $$\int \frac{x^2}{(x-1)(x^2+2x+1)} dx = \int \frac{x^2}{(x-1)(x+1)^2} dx$$

$$= \int \left[\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \right] dx$$

$$\Big(A(x+1)^2 + B(x-1)(x+1) + C(x-1) = x^2$$

Let $x=1$: $4A = 1 \rightarrow A = 1/4$

Let $x=-1$: $-2C = 1 \rightarrow C = -1/2$

Let $x=0$: $\frac{1}{4} - B + \frac{1}{2} = 0 \rightarrow B = 3/4$ $\Big)$

$$= \int \left[\frac{1/4}{x-1} + \frac{3/4}{x+1} + \frac{-1/2}{(x+1)^2} \right] dx$$

$$= \frac{1}{4}\cdot \ln|x-1| + \frac{3}{4}\cdot \ln|x+1| + \frac{1}{2}\cdot \frac{1}{x+1} + C$$

21.) $$\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \int_0^1 \left[\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right] dx$$

$$\left(A(x^2+1) + (Bx+C)(x+1) = 1\right.$$

Let $x=-1$: $2A = 1 \longrightarrow A = 1/2$

Let $x = i$: $(Bi+C)(i+1) = 1 \longrightarrow$

$Bi^2 + Bi + Ci + C = 1 \longrightarrow$

$(B+C)i + (C-B) = (0)\cdot i + (1) \longrightarrow$

$\underline{B+C=0}$ and $\underline{C-B=1} \longrightarrow$

$B + (B+1) = 0 \longrightarrow 2B+1=0 \longrightarrow B = -1/2$

and $\left. C = 1/2 \right)$

$$= \int_0^1 \left[\frac{1/2}{x+1} + \frac{-1/2\,x + 1/2}{x^2+1}\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{x+1} + \frac{-x}{x^2+1} + \frac{1}{x^2+1}\right] dx$$

$$= \frac{1}{2}\left(\ln|x+1| - \frac{1}{2}\ln|x^2+1| + \arctan x\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\ln 2 - \frac{1}{2}\ln 2 + \arctan 1\right)$$

$$- \frac{1}{2}\left(\cancel{\ln 1}^{\,0} - \frac{1}{2}\cancel{\ln 1}^{\,0} + \cancel{\arctan 0}^{\,0}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\ln 2 + \frac{\pi}{4}\right) = \frac{1}{4}\cdot \ln 2 + \frac{\pi}{8}$$

23.) $$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \left[\frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2}\right] dy$$

$$\left((Ay+B)(y^2+1) + (Cy+D) = y^2+2y+1\right.$$

Let $y=i$: $Ci + D = i^2 + 2i + 1 \longrightarrow$

$Ci + D = 2i + 0 \rightarrow C=2 \,,\; D=0$

Let $y=0$: $B=1$

Let $y=1$: $(A+1)(2)+2=4 \rightarrow A=0$)

$$= \int \left[\frac{1}{y^2+1} + \frac{2y}{(y^2+1)^2} \right] dy$$

$$= \arctan y + \frac{-1}{y^2+1} + C$$

26.) $$\int \frac{s^4+81}{s(s^2+9)^2}\, ds$$

$$= \int \left[\frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \right] ds$$

($A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s = s^4+81$

Let $s=0$: $81A = 81 \rightarrow A=1$

Let $s=3i$: $(D\cdot 3i + E)\cdot(3i) = (3i)^4 + 81 \rightarrow$

$9Di^2 + 3Ei = 81+81 \rightarrow (3E)i + (-9D) = (0)i + 162$

$\rightarrow 3E=0 \rightarrow E=0$ and $-9D = 162 \rightarrow D=-18$

Let $s=1$: $100 + (B+C)(10) - 18 = 82 \rightarrow$

$10(B+C) = 0 \rightarrow \underline{B+C=0}$

Let $s=-1$: $100 + (C-B)(-10) - 18 = 82 \rightarrow$

$-10(C-B) = 0 \rightarrow \underline{C-B=0}$; then

$2C = 0 \rightarrow C=0$ and $B=0$)

$$= \int \left[\frac{1}{s} + \frac{-18s}{(s^2+9)^2} \right] ds$$

$$= \ln|s| - 18\cdot\left(\frac{-1}{2}\right)\cdot\frac{1}{s^2+9} + C$$

$$= \ln|s| + \frac{9}{s^2+9} + C$$

34.)

$$\begin{array}{r|l} & x^2+1 \\ \hline x^2-1 & x^4 \\ & -(x^4-x^2) \\ \hline & x^2 \\ & -(x^2-1) \\ \hline & 1 \end{array}$$

$$\int \frac{x^4}{x^2-1}\,dx = \int \left[x^2+1 + \frac{1}{x^2-1}\right] dx$$

$$= \frac{x^3}{3} + x + \int \frac{1}{(x-1)(x+1)}\,dx$$

$$= \frac{x^3}{3} + x + \int \left[\frac{A}{x-1} + \frac{B}{x+1}\right] dx$$

$$\Big(A(x+1) + B(x-1) = 1$$

Let $x = 1$: $2A = 1 \longrightarrow A = 1/2$

Let $x = -1$: $-2B = 1 \longrightarrow B = -1/2$ $\Big)$

$$= \frac{x^3}{3} + x + \int \left[\frac{1/2}{x-1} + \frac{-1/2}{x+1}\right] dx$$

$$= \frac{x^3}{3} + x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

37.)

$$\begin{array}{r|l} & y \\ \hline y^3+y & y^4+y^2-1 \\ & -(y^4+y^2) \\ \hline & -1 \end{array}$$

$$\int \frac{y^4+y^2-1}{y^3+y}\,dy = \int \left[y - \frac{1}{y\cdot(y^2+1)}\right] dy$$

$$= \frac{y^2}{2} - \int \left[\frac{A}{y} + \frac{By+C}{y^2+1}\right] dy$$

$\big(A(y^2+1) + (By+C)y = 1$

Let $y=0$: $A = 1$

Let $y=i$: $(Bi+C)i = 1 \to$

$Bi^2 + Ci = 1 \to (C)i + (-B) = (0)i + (1) \to$

$C=0$ and $-B = 1 \to B = -1 \big)$

$$= \frac{y^2}{2} - \int \left[\frac{1}{y} + \frac{-y}{y^2+1} \right] dy$$

$$= \frac{y^2}{2} - \left(\ln|y| - \frac{1}{2}\ln|y^2+1| \right) + c$$

$$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln|y^2+1| + c$$

39.) $\displaystyle\int \frac{e^t}{e^{2t}+3e^t+2}\,dt = \int \frac{e^t}{(e^t)^2+3(e^t)+2}\,dt$

(Let $u = e^t \to du = e^t\,dt$)

$$= \int \frac{1}{u^2+3u+2}\,du = \int \frac{1}{(u+1)(u+2)}\,du$$

$$= \int \left[\frac{A}{u+1} + \frac{B}{u+2} \right] du$$

$\big(A(u+2) + B(u+1) = 1$

Let $u=-2$: $-B = 1 \to B = -1$

Let $u=-1$: $A = 1 \big)$

$$= \int \left[\frac{1}{u+1} + \frac{-1}{u+2} \right] du = \ln|u+1| - \ln|u+2| + c$$

$$= \ln|e^t+1| - \ln|e^t+2| + c$$

42.) $\displaystyle\int \frac{\sin\theta}{\cos^2\theta + \cos\theta - 2}\,d\theta = \int \frac{\sin\theta}{(\cos\theta)^2 + (\cos\theta) - 2}\,d\theta$

(Let $u = \cos\theta \rightarrow du = -\sin\theta\, d\theta$
$\rightarrow -du = \sin\theta\, d\theta$)

$$= -\int \frac{1}{u^2+u-2}\, du = -\int \frac{1}{(u-1)(u+2)}\, du$$

$$= -\int \left[\frac{A}{u-1} + \frac{B}{u+2}\right] du$$

($A(u+2) + B(u-1) = 1$
Let $u=1$: $3A = 1 \longrightarrow A = 1/3$
Let $u=-2$: $-3B = 1 \longrightarrow B = -1/3$)

$$= -\int \left[\frac{1/3}{u-1} + \frac{-1/3}{u+2}\right] du$$

$$= -\left[\frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+2|\right] + C$$

$$= -\frac{1}{3}\ln|\cos\theta - 1| + \frac{1}{3}\ln|\cos\theta + 2| + C$$

45.) $\int \frac{1}{x^{3/2} - x^{1/2}}\, dx = \int \frac{1}{x^{1/2}(x-1)}\, dx$

$$= \int \frac{1}{x^{1/2}((x^{1/2})^2 - 1)}\, dx$$ (Let $u = x^{1/2} \xrightarrow{D}$
$du = \frac{1}{2}x^{-1/2}\, dx \rightarrow 2\, du = \frac{1}{x^{1/2}}\, dx$)

$$= 2\int \frac{1}{u^2-1}\, du = 2\int \frac{1}{(u-1)(u+1)}\, du$$

$$= 2\int \left[\frac{A}{u-1} + \frac{B}{u+1}\right] du$$ ($A(u+1) + B(u-1) = 1$
Let $u=1$: $2A = 1 \rightarrow A = 1/2$
Let $u=-1$: $-2B = 1 \rightarrow B = -1/2$)

$$= 2\int \left[\frac{1/2}{u-1} + \frac{-\frac{1}{2}}{u+1}\right] du$$

$$= 2\left(\tfrac{1}{2}\ln|u-1| - \tfrac{1}{2}\ln|u+1|\right) + C$$

$$= \ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + C$$

47.) $\int \frac{\sqrt{x+1}}{x}\,dx$ (Let $u^2 = x+1 \xrightarrow{D} 2u\,du = dx$)

and $x = u^2 - 1$

$u = \sqrt{x+1}$

$$= \int \frac{u}{u^2-1}\, 2u\,du$$

$$= 2\int \frac{u^2}{u^2-1}\,du$$

$$\begin{array}{r} 1 \\ u^2-1 \overline{\smash{)}\,u^2 } \\ -(u^2-1) \\ \hline 1 \end{array}$$

$$= 2\int \left[1 + \frac{1}{u^2-1}\right] du$$

$$= 2\left(u + \int \frac{1}{u^2-1}\,du\right) = 2\sqrt{x+1} + 2\int \frac{1}{(u-1)(u+1)}\,du$$

$$= 2\sqrt{x+1} + 2\int \left[\frac{A}{u-1} + \frac{B}{u+1}\right] du$$

$= $ (SEE prob. 45) $2\sqrt{x+1} + 2\int\left[\frac{\frac{1}{2}}{u-1} + \frac{-\frac{1}{2}}{u+1}\right] du$

$$= 2\sqrt{x+1} + 2\left(\tfrac{1}{2}\ln|u-1| - \tfrac{1}{2}\ln|u+1|\right) + C$$

$$= 2\sqrt{x+1} + \ln|\sqrt{x+1}-1| - \ln|\sqrt{x+1}+1| + C$$

48.) $\int \frac{1}{x\sqrt{x+9}}\,dx$ (Let $u^2 = x+9 \xrightarrow{D}$ $2u\,du = dx$ and $x = u^2-9$, $u = \sqrt{x+9}$)

$$= \int \frac{2\cancel{u}}{(u^2-9)\cdot\cancel{u}}\,du = \int \frac{2}{(u-3)(u+3)}\,du$$

$$= \int \left[\frac{A}{u-3} + \frac{B}{u+3}\right] du \quad (A(u+3)+B(u-3)=2 \rightarrow$$

Let $u=3$: $6A=2 \rightarrow A = 1/3$

Let $u=-3$: $-6B = 2 \rightarrow B = -1/3$)

$$= \int \left[\frac{1/3}{u-3} + \frac{-1/3}{u+3}\right] du$$

$$= \frac{1}{3}\ln|u-3| - \frac{1}{3}\ln|u+3| + c$$

$$= \frac{1}{3}\ln\left|\sqrt{x+9}-3\right| - \frac{1}{3}\ln\left|\sqrt{x+9}+3\right| + c$$

49.) $\int \frac{1}{x(x^4+1)}dx = \int \frac{x^3}{x^4(x^4+1)}dx$

(Let $u = x^4+1 \xrightarrow{D} du = 4x^3dx \rightarrow \frac{1}{4}du = x^3dx$)

$\hookrightarrow$ and $x^4 = u-1$

$$= \frac{1}{4}\int \frac{1}{(u-1)u}du = \frac{1}{4}\int\left[\frac{A}{u-1} + \frac{B}{u}\right]du$$

($Au + B(u-1) = 1$:

Let $u=0$: $-B = 1 \rightarrow B = -1$

Let $u=1$: $A = 1$)

$$= \frac{1}{4}\int\left[\frac{1}{u-1} + \frac{-1}{u}\right]du$$

$$= \frac{1}{4}\left(\ln|u-1| - \ln|u|\right) + c$$

$$= \frac{1}{4}\left(\ln|(x^4+1)-1| - \ln|x^4+1|\right) + c$$

$$= \frac{1}{4}\ln x^4 - \frac{1}{4}\ln|x^4+1| + c$$

$= \ln x - \frac{1}{4}\ln|x^4+1| + C$

50.) $\int \frac{1}{x^6(x^5+4)}dx = \int \frac{1}{x\cdot x^5(x^5+4)}dx$

$= \int \frac{x^4}{x^5x^5(x^5+4)}dx$ (Let $u = x^5 \rightarrow$
$du = 5x^4dx \rightarrow \frac{1}{5}du = x^4dx$)

$= \frac{1}{5}\int \frac{1}{u^2(u+1)}du$

$= \frac{1}{5}\int \left[\frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}\right]du$

($Au(u+1) + B(u+1) + Cu^2 = 1$
Let $u=0$: $B=1$
Let $u=-1$: $C=1$
Let $u=1$: $2A + 2(1) + (1)(1) = 1$
$\rightarrow 2A = -2 \rightarrow A = -1$)

$= \frac{1}{5}\int \left[\frac{-1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\right]du$

$= \frac{1}{5}\left(-\ln|u| - \frac{1}{u} + \ln|u+1|\right) + C$

$= \frac{1}{5}\left(-\ln|x^5| - \frac{1}{x^5} + \ln|x^5+1|\right) + C$

55.) (DISC METHOD)

$$Vol = \pi \int_{1/2}^{5/2} (\text{radius})^2 \, dx = \pi \int_{1/2}^{5/2} \left(\frac{3}{\sqrt{3x - x^2}} \right)^2 dx$$

$$= \pi \int_{1/2}^{5/2} \frac{9}{3x - x^2} \, dx = 9\pi \int_{1/2}^{5/2} \frac{1}{x(3-x)} \, dx$$

$$= 9\pi \int_{1/2}^{5/2} \left[\frac{A}{x} + \frac{B}{3-x} \right] dx$$

($A(3-x) + Bx = 1$

Let $x = 3$: $3B = 1 \rightarrow B = 1/3$

Let $x = 0$: $3A = 1 \rightarrow A = 1/3$)

$$= 9\pi \int_{\frac{1}{2}}^{\frac{5}{2}} \left[\frac{1/3}{x} + \frac{1/3}{3-x} \right] dx$$

$$= 9\pi \left[\frac{1}{3} \ln|x| - \frac{1}{3} \ln|3-x| \right] \Big|_{1/2}^{5/2}$$

$$= 9\pi \left(\frac{1}{3} \ln \frac{5}{2} - \frac{1}{3} \ln \frac{1}{2} \right)$$

$$- 9\pi \left(\frac{1}{3} \ln \frac{1}{2} - \frac{1}{3} \ln \frac{5}{2} \right)$$

$$= 9\pi \left(\frac{1}{3} \right) \left(\ln \frac{5}{2} - \ln \frac{1}{2} - \ln \frac{1}{2} + \ln \frac{5}{2} \right)$$

$$= 3\pi \left(2 \ln \frac{5}{2} - 2 \ln 1/2 \right)$$

$$= 6\pi \left(\ln 5/2 - \ln 1/2 \right)$$

$$= 6\pi \cdot \ln \left(\frac{5/2}{1/2} \right)$$

$$= 6\pi \ln 5$$