

## Section 8.7

$$\begin{aligned}
 1.) \int_0^{\infty} \frac{1}{x^2+1} dx &= \lim_{A \rightarrow \infty} \int_0^A \frac{1}{x^2+1} dx \\
 &= \lim_{A \rightarrow \infty} \arctan x \Big|_0^A \\
 &= \lim_{A \rightarrow \infty} (\arctan A - \arctan 0) = \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 4.) \int_0^4 \frac{1}{\sqrt{4-x}} dx &= \lim_{A \rightarrow 4^-} \int_0^A \frac{1}{\sqrt{4-x}} dx \\
 &= \lim_{A \rightarrow 4^-} \int_0^A (4-x)^{-1/2} dx \\
 &= \lim_{A \rightarrow 4^-} -2(4-x)^{1/2} \Big|_0^A \\
 &= \lim_{A \rightarrow 4^-} \{-2\sqrt{4-A} - -2\sqrt{4}\} = 0 + 4 = 4
 \end{aligned}$$

$$\begin{aligned}
 6.) \int_{-8}^1 \frac{1}{x^{1/3}} dx &= \int_{-8}^0 \frac{1}{x^{1/3}} dx + \int_0^1 \frac{1}{x^{1/3}} dx \\
 &= C + D :
 \end{aligned}$$

$$\begin{aligned}
 C &= \lim_{A \rightarrow 0^-} \int_{-8}^A x^{-1/3} dx = \lim_{A \rightarrow 0^-} \frac{3}{2} x^{2/3} \Big|_{-8}^A \\
 &= \lim_{A \rightarrow 0^-} \left( \frac{3}{2} A^{2/3} - \frac{3}{2} (-8)^{2/3} \right) = 0 - \frac{3}{2} (4) = -6 ;
 \end{aligned}$$

$$\begin{aligned}
 D &= \lim_{A \rightarrow 0^+} \int_A^1 x^{-1/3} dx = \lim_{A \rightarrow 0^+} \frac{3}{2} x^{2/3} \Big|_A^1 \\
 &= \lim_{A \rightarrow 0^+} \left( \frac{3}{2} (1)^{2/3} - \frac{3}{2} A^{2/3} \right) = \frac{3}{2} - 0 = \frac{3}{2} ;
 \end{aligned}$$

$$\text{then } \int_{-8}^1 \frac{1}{x^{1/3}} dx = C + D = -6 + \frac{3}{2} = -\frac{9}{2} .$$

$$\begin{aligned}
9.) \int_{-\infty}^{-2} \frac{2}{x^2-1} dx &= \int_{-\infty}^{-2} \frac{2}{(x-1)(x+1)} dx \\
&= \int_{-\infty}^{-2} \left[ \frac{A}{x-1} + \frac{B}{x+1} \right] dx = \int_{-\infty}^{-2} \left[ \frac{1}{x-1} + \frac{-1}{x+1} \right] dx \\
&= \lim_{A \rightarrow -\infty} \int_A^{-2} \left[ \frac{1}{x-1} - \frac{1}{x+1} \right] dx \\
&= \lim_{A \rightarrow -\infty} \left( \ln|x-1| - \ln|x+1| \right) \Big|_A^{-2} \\
&= \lim_{A \rightarrow -\infty} \ln \left| \frac{x-1}{x+1} \right| \Big|_A^{-2} \\
&= \lim_{A \rightarrow -\infty} \left\{ \ln 3 - \ln \left| \frac{A-1}{A+1} \right| \right\} \\
&= \ln 3 - \lim_{A \rightarrow -\infty} \ln \left| \frac{1 - \frac{1}{A}}{1 + \frac{1}{A}} \right| \\
&= \ln 3 - \ln \left| \frac{1-0}{1+0} \right| = \ln 3 - \ln 1 = \ln 3
\end{aligned}$$

$$\begin{aligned}
14.) \int_{-\infty}^{\infty} \frac{x}{(x^2+4)^{3/2}} dx &= \int_{-\infty}^0 \frac{x}{(x^2+4)^{3/2}} dx \\
&+ \int_0^{\infty} \frac{x}{(x^2+4)^{3/2}} dx = C + D ; \\
C &= \lim_{A \rightarrow -\infty} \int_A^0 x \cdot (x^2+4)^{-3/2} dx \\
&= \lim_{A \rightarrow -\infty} \frac{1}{2} \cdot \frac{(x^2+4)^{-1/2}}{-1/2} \Big|_A^0 \\
&= \lim_{A \rightarrow -\infty} \left\{ -(4)^{-1/2} - -(A^2+4)^{-1/2} \right\} \\
&= \lim_{A \rightarrow -\infty} \left\{ -\frac{1}{2} + \frac{1}{\sqrt{A^2+4}} \right\} = -\frac{1}{2} + 0 = -\frac{1}{2} ;
\end{aligned}$$

$$D = \lim_{A \rightarrow \infty} \int_0^A x(x^2+4)^{-3/2} dx$$

$$= \lim_{A \rightarrow \infty} \left. \frac{-1}{\sqrt{x^2+4}} \right|_0^A = \lim_{A \rightarrow \infty} \left\{ \frac{-1}{\sqrt{A^2+4}} - \frac{-1}{\sqrt{4}} \right\}$$

$$= 0 + \frac{1}{2} = \frac{1}{2}; \text{ then}$$

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+4)^{3/2}} dx = C + D = \frac{-1}{2} + \frac{1}{2} = 0$$

$$15.) \int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}} d\theta = \lim_{A \rightarrow 0^+} \int_A^1 (\theta+1)(\theta^2+2\theta)^{-1/2} d\theta$$

$$= \lim_{A \rightarrow 0^+} \left. (\theta^2+2\theta)^{1/2} \right|_A^1 = \lim_{A \rightarrow 0^+} (\sqrt{3} - \sqrt{A^2+2A})$$

$$= \sqrt{3} - 0 = \sqrt{3}$$

$$20.) \int_0^{\infty} \frac{16 \arctan x}{1+x^2} dx$$

$$= \lim_{A \rightarrow \infty} \int_0^A \frac{16 \arctan x}{1+x^2} dx$$

$$= \lim_{A \rightarrow \infty} 16 \cdot \frac{1}{2} (\arctan x)^2 \Big|_0^A$$

$$= \lim_{A \rightarrow \infty} \left\{ 8 (\arctan x)^2 - 8 (\arctan 0)^2 \right\}$$

$$= 8 \left( \frac{\pi}{2} \right)^2 - 8(0)^2 = 2\pi^2$$

$$21.) \int_{-\infty}^0 \theta e^{\theta} d\theta \quad (\text{Let } u = \theta, \quad dv = e^{\theta} d\theta \\ du = d\theta, \quad v = e^{\theta})$$

$$= \lim_{A \rightarrow -\infty} \int_A^0 \theta e^{\theta} d\theta$$

$$= \lim_{A \rightarrow -\infty} \left\{ e^{\theta} \Big|_A^0 - \int_A^0 e^{\theta} d\theta \right\}$$

$$= \lim_{A \rightarrow -\infty} \left\{ 0 - Ae^A - e^{\theta} \Big|_A^0 \right\}$$

$$= \lim_{A \rightarrow -\infty} \left\{ -Ae^A - (1 - e^A) \right\}$$

↑ "∞ · 0" (indeterminate)

$$= \lim_{A \rightarrow -\infty} \left\{ \frac{-A}{e^{-A}} - 1 + e^A \right\}$$

↑ "∞ / ∞" (L'Hopital)

$$= \lim_{A \rightarrow -\infty} \left\{ \frac{-1}{-e^{-A}} - 1 + e^A \right\}$$

$$= \lim_{A \rightarrow -\infty} \left\{ 2e^A - 1 \right\} = 2e^{-\infty} - 1$$

$$= 2 \cdot \frac{1}{e^{\infty}} - 1 = 2(0) - 1 = -1$$

$$25.) \int_0^1 x \ln x \, dx = \lim_{A \rightarrow 0^+} \int_A^1 x \ln x \, dx$$

$$\left( \text{Let } u = \ln x, \, dv = x \, dx \right. \\ \left. du = \frac{1}{x} \, dx, \, v = \frac{1}{2} x^2 \right)$$

$$= \lim_{A \rightarrow 0^+} \left\{ \frac{1}{2} x^2 \ln x \Big|_A^1 - \frac{1}{2} \int_A^1 x \, dx \right\}$$

$$= \lim_{A \rightarrow 0^+} \left\{ \frac{1}{2} (1)^2 \ln 1 - \frac{1}{2} A^2 \ln A - \frac{1}{2} \cdot \frac{x^2}{2} \Big|_A^1 \right\}$$

$$= \lim_{A \rightarrow 0^+} \left\{ -\frac{1}{2} A^2 \ln A - \left( \frac{1}{4} - \frac{A^2}{4} \right) \right\}$$

↑ "0 · ∞" (indeterminate)

$$= \lim_{A \rightarrow 0^+} \left\{ \frac{-\ln A}{2/A^2} - \frac{1}{4} + \frac{A^2}{4} \right\}$$

↖ "∞/∞" (L'Hopital)

$$= \lim_{A \rightarrow 0^+} \left\{ \frac{-1/A}{-4/A^3} - \frac{1}{4} + \frac{A^2}{4} \right\}$$

$$= \lim_{A \rightarrow 0^+} \left\{ \frac{1}{2} A^2 - \frac{1}{4} \right\} = 0 - \frac{1}{4} = -\frac{1}{4}$$

$$30.) \int_2^4 \frac{1}{t\sqrt{t^2-2^2}} dt = \lim_{A \rightarrow 2^+} \int_A^4 \frac{1}{t\sqrt{t^2-2^2}} dt$$

$$= \lim_{A \rightarrow 2^+} \frac{1}{2} \operatorname{arcsec} \left( \frac{t}{2} \right) \Big|_A^4$$

$$= \lim_{A \rightarrow 2^+} \left\{ \frac{1}{2} \operatorname{arcsec}(2) - \frac{1}{2} \operatorname{arcsec} \left( \frac{A}{2} \right) \right\}$$

$$= \frac{1}{2} \cdot \left( \frac{\pi}{3} \right) - \frac{1}{2} (0) = \frac{\pi}{6}$$

$$34.) \int_0^{\infty} \frac{1}{(x+1)(x^2+1)} dx = \int_0^{\infty} \left[ \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \right] dx$$

$$\left\{ \begin{aligned} A(x^2+1) + (Bx+C)(x+1) &= 1 \end{aligned} \right.$$

$$\text{Let } x = -1: 2A = 1 \rightarrow A = \frac{1}{2}$$

$$\text{Let } x = i: (Bi+C)(i+1) = 1 \rightarrow$$

$$-B + Ci + Bi + C = 1 \rightarrow$$

$$(B+C)i + (C-B) = (0)i + (1) \rightarrow$$

$$\left. \begin{aligned} B+C &= 0 \\ C-B &= 1 \end{aligned} \right\} \rightarrow 2C = 1 \rightarrow C = \frac{1}{2},$$

$$\left. \begin{aligned} B &= -\frac{1}{2} \end{aligned} \right\}$$

$$= \int_0^{\infty} \left[ \frac{1/2}{x+1} + \frac{-1/2x + 1/2}{x^2+1} \right] dx$$

$$= \lim_{A \rightarrow \infty} \left\{ \frac{1}{2} \ln|x+1| - \frac{1}{4} \ln|x^2+1| + \frac{1}{2} \arctan x \right\} \Big|_0^A$$

$$= \lim_{A \rightarrow \infty} \left\{ \frac{1}{2} \ln|A+1| - \frac{1}{4} \ln|A^2+1| + \frac{1}{2} \arctan A \right. \\ \left. - \left( \frac{1}{2} \ln 1 - \frac{1}{4} \ln 1 + \frac{1}{2} \arctan 0 \right) \right\}$$

$$= \lim_{A \rightarrow \infty} \left\{ \frac{1}{2} \ln(A+1) - \frac{1}{2} \cdot \frac{1}{2} \ln(A^2+1) \right\} + \frac{1}{2} \left( \frac{\pi}{2} \right)$$

$$= \lim_{A \rightarrow \infty} \left\{ \frac{1}{2} \ln(A+1) - \frac{1}{2} \ln \sqrt{A^2+1} \right\} + \frac{\pi}{4}$$

$$= \lim_{A \rightarrow \infty} \frac{1}{2} \ln \left( \frac{A+1}{\sqrt{A^2+1}} \right) + \frac{\pi}{4}$$

$$= \lim_{A \rightarrow \infty} \frac{1}{2} \ln \left( \frac{A+1}{\sqrt{A^2(1+1/A^2)}} \right) + \frac{\pi}{4}$$

$$= \lim_{A \rightarrow \infty} \frac{1}{2} \ln \left( \frac{A+1}{A} \cdot \frac{1}{\sqrt{1+1/A^2}} \right) + \frac{\pi}{4}$$

$$= \lim_{A \rightarrow \infty} \frac{1}{2} \ln \left( \left(1 + \frac{1}{A}\right) \cdot \frac{1}{\sqrt{1+1/A^2}} \right) + \frac{\pi}{4}$$

$$= \frac{1}{2} \cdot \ln 1 + \frac{\pi}{4} = \frac{\pi}{4}$$

$$37.) 0 \leq \frac{\sin \theta}{\sqrt{\pi - \theta}} \leq \frac{1}{\sqrt{\pi - \theta}} \quad \text{so}$$

$$\int_0^{\pi} \frac{\sin \theta}{\sqrt{\pi - \theta}} d\theta \leq \int_0^{\pi} \frac{1}{\sqrt{\pi - \theta}} d\theta$$

$$= \lim_{A \rightarrow \pi^-} \int_0^A \frac{1}{\sqrt{\pi - \theta}} d\theta = \lim_{A \rightarrow \pi^-} \int_0^A (\pi - \theta)^{-1/2} d\theta$$

$$\begin{aligned}
&= \lim_{A \rightarrow \pi^-} \left. - \frac{(\pi - \theta)^{1/2}}{1/2} \right|_0^A \\
&= \lim_{A \rightarrow \pi^-} \left\{ -2\sqrt{\pi - A} - (-2\sqrt{\pi}) \right\} \\
&= -2(0) + 2\sqrt{\pi} = 2\sqrt{\pi}; \text{ so} \\
&\int_0^\pi \frac{\sin \theta}{\sqrt{\pi - \theta}} d\theta \text{ converges}
\end{aligned}$$

$$\begin{aligned}
41.) \quad 0 &\leq \frac{1}{\sqrt{t} + \sin t} \leq \frac{1}{\sqrt{t}} \text{ so} \\
\int_0^\pi \frac{1}{\sqrt{t} + \sin t} dt &\leq \int_0^\pi \frac{1}{\sqrt{t}} dt \\
&= \lim_{A \rightarrow 0^+} \int_A^\pi t^{-1/2} dt = \lim_{A \rightarrow 0^+} 2\sqrt{t} \Big|_A^\pi \\
&= \lim_{A \rightarrow 0^+} (2\sqrt{\pi} - 2\sqrt{A}) = 2\sqrt{\pi} - 0 = 2\sqrt{\pi}
\end{aligned}$$

$$\begin{aligned}
42.) \quad 0 &\leq t - \sin t \leq t \rightarrow \\
0 &\leq \frac{1}{t} \leq \frac{1}{t - \sin t} \rightarrow
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{1}{t - \sin t} dt &\geq \int_0^1 \frac{1}{t} dt = \lim_{A \rightarrow 0^+} \int_A^1 \frac{1}{t} dt \\
&= \lim_{A \rightarrow 0^+} \ln|t| \Big|_A^1 = \lim_{A \rightarrow 0^+} (\ln 1 - \ln A) \\
&= -(-\infty) = \infty; \text{ so} \\
\int_0^1 \frac{1}{t - \sin t} dt &\text{ diverges.}
\end{aligned}$$

$$51.) \int_0^{\infty} \frac{1}{\sqrt{x^6+1}} dx = \int_0^1 \frac{1}{\sqrt{x^6+1}} dx + \int_1^{\infty} \frac{1}{\sqrt{x^6+1}} dx$$

$$= C + D ;$$

$$C = \int_0^1 \frac{1}{\sqrt{x^6+1}} dx \leq \int_0^1 \frac{1}{\sqrt{0+1}} dx = \int_0^1 1 dx$$

$$= x \Big|_0^1 = 1 \quad \text{so } \int_0^1 \frac{1}{\sqrt{x^6+1}} dx \text{ converges;}$$

$$D = \int_1^{\infty} \frac{1}{\sqrt{x^6+1}} dx \leq \int_1^{\infty} \frac{1}{\sqrt{x^6+0}} dx$$

$$= \int_1^{\infty} \frac{1}{x^3} dx = \lim_{A \rightarrow \infty} \left. \frac{-1}{2x^2} \right|_1^A$$

$$= \lim_{A \rightarrow \infty} \left( \frac{-1}{2A^2} - \frac{-1}{2} \right) = 0 + \frac{1}{2} = \frac{1}{2}$$

so  $\int_1^{\infty} \frac{1}{\sqrt{x^6+1}} dx$  converges; then

$$\int_0^{\infty} \frac{1}{\sqrt{x^6+1}} dx \text{ converges.}$$

$$53.) \int_1^{\infty} \frac{\sqrt{x+1}}{x^2} dx \leq \int_1^{\infty} \frac{\sqrt{x+x}}{x^2} dx$$

$$= \lim_{A \rightarrow \infty} \int_1^A \frac{\sqrt{2} \cdot \sqrt{x}}{x^2} dx = \lim_{A \rightarrow \infty} \sqrt{2} \int_1^A \frac{1}{x^{3/2}} dx$$

$$= \lim_{A \rightarrow \infty} \sqrt{2} \left. \frac{x^{-1/2}}{-1/2} \right|_1^A = \lim_{A \rightarrow \infty} -2\sqrt{2} \left( \frac{1}{\sqrt{A}} - \frac{1}{\sqrt{1}} \right)$$

$$= -2\sqrt{2} (0 - 1) = 2\sqrt{2}, \quad \text{so}$$

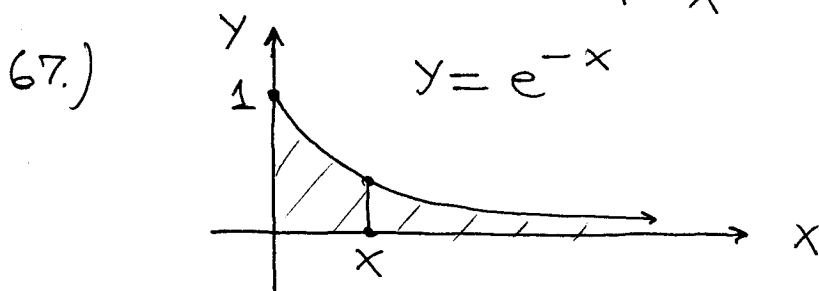
$$\int_1^{\infty} \frac{\sqrt{x+1}}{x^2} dx \text{ converges.}$$



59.)  $0 \leq \frac{1}{x} \leq \frac{e^x}{x}$  for  $x \geq 1$  so

$$\int_1^{\infty} \frac{e^x}{x} dx \geq \int_1^{\infty} \frac{1}{x} dx = \lim_{A \rightarrow \infty} \int_1^A \frac{1}{x} dx$$
$$= \lim_{A \rightarrow \infty} \ln x \Big|_1^A = \lim_{A \rightarrow \infty} (\ln A - \ln 1)$$

=  $\infty$ ; then  $\int_1^{\infty} \frac{e^x}{x} dx$  diverges.



$$\text{Area} = \int_0^{\infty} e^{-x} dx = \lim_{A \rightarrow \infty} \int_0^A e^{-x} dx$$

$$= \lim_{A \rightarrow \infty} -e^{-x} \Big|_0^A = \lim_{A \rightarrow \infty} \left( -\frac{1}{e^A} - -e^0 \right)$$

$$= 0 + 1 = 1$$

69

NEXT  
PAGE

70.) (DISC METHOD)

$$\text{Vol} = \pi \int_0^{\infty} (\text{radius})^2 dx$$

$$= \pi \int_0^{\infty} (e^{-x})^2 dx$$

$$= \lim_{A \rightarrow \infty} \pi \int_0^A e^{-2x} dx = \lim_{A \rightarrow \infty} \pi \cdot \frac{e^{-2x}}{-2} \Big|_0^A$$

$$= \lim_{A \rightarrow \infty} -\frac{\pi}{2} \left( \frac{1}{e^{2A}} - e^0 \right) = -\frac{\pi}{2} (0 - 1)$$

$$= \frac{\pi}{2}$$

69.) (SHELL METHOD)

$$\text{Vol} = 2\pi \int_0^{\infty} (\text{radius})(\text{height}) dx$$

$$= 2\pi \int_0^{\infty} (x)(e^{-x}) dx$$

$$= \lim_{A \rightarrow \infty} 2\pi \int_0^A x e^{-x} dx$$

$$\left( \text{Let } u = x, \quad dv = e^{-x} dx \right. \\ \left. du = dx, \quad v = -e^{-x} \right)$$

$$= \lim_{A \rightarrow \infty} 2\pi \left\{ -x e^{-x} \Big|_0^A - \int_0^A e^{-x} dx \right\}$$

$$= \lim_{A \rightarrow \infty} 2\pi \left\{ \frac{-A}{e^A} - e^{-x} \Big|_0^A \right\}$$

$$= \lim_{A \rightarrow \infty} 2\pi \left\{ \frac{-A}{e^A} - \left( \frac{1}{e^A} - 1 \right) \right\}$$

$\sim \frac{\infty}{\infty}$  (L'Hopital)

$$= \lim_{A \rightarrow \infty} 2\pi \left\{ \frac{-1}{e^A} - \frac{1}{e^A} + 1 \right\}$$

$$= 2\pi \left\{ 0 - 0 + 1 \right\}$$

$$= 2\pi$$

75.) b.) Show that  $\int_0^{\infty} \frac{\sin t}{t} dt < \infty$  :

$$\int_0^{\infty} \frac{\sin t}{t} dt = \int_0^1 \frac{\sin t}{t} dt + \int_1^{\infty} \frac{\sin t}{t} dt$$
$$= B + C ;$$

by MVT applied to  $f(x) = \sin x$  on  $[0, t]$   
we get  $\frac{f(t) - f(0)}{t - 0} = f'(c)$  for  $0 < c < t$

$$\rightarrow \frac{\sin t - \sin 0}{t} = \cos c < 1$$

$$\rightarrow \frac{\sin t}{t} \leq 1 ; \text{ thus,}$$

$$B = \int_0^1 \frac{\sin t}{t} dt \leq \int_0^1 1 dt = t \Big|_0^1 = 1 < \infty.$$

$$\text{For } C = \int_1^{\infty} \frac{\sin t}{t} dt$$

$$(\text{Let } u = 1/t, \quad dv = \sin t dt$$

$$\rightarrow du = -\frac{1}{t^2} dt, \quad v = -\cos t)$$

$$= -\frac{\cos t}{t} \Big|_1^{\infty} - \int_1^{\infty} \frac{\cos t}{t^2} dt ;$$

$$= -\frac{\cos t}{t} \Big|_1^{\infty} = \lim_{A \rightarrow \infty} -\frac{\cos t}{t} \Big|_1^A$$

$$= \lim_{A \rightarrow \infty} -\frac{\cos A}{A} - \frac{-\cos 1}{1}$$

by Squeeze Principle

$$= 0 + \cos 1 < \infty ;$$

$$\int_1^{\infty} \left| \frac{\cos t}{t^2} \right| dt = \int_1^{\infty} \frac{|\cos t|}{t^2} dt$$

$$\leq \int_1^{\infty} \frac{1}{t^2} dt = \lim_{A \rightarrow \infty} \int_1^A \frac{1}{t^2} dt$$

$$= \lim_{A \rightarrow \infty} \left. -\frac{1}{t} \right|_1^A = \lim_{A \rightarrow \infty} \left( -\frac{1}{A} - \frac{-1}{1} \right)$$

$$= 0 + 1 < \infty ; \text{ then}$$

$$C = \int_1^{\infty} \frac{\sin t}{t} dt = -\frac{\cos t}{t} \Big|_1^{\infty} - \int_1^{\infty} \frac{\cos t}{t^2} dt$$

$$< \infty , \text{ and}$$

$$\int_0^{\infty} \frac{\sin t}{t} dt = B + C < \infty .$$