Section 5.6

2.) a.) $\int_0^1 r\sqrt{1-r^2}\,dr$ (Let $u = 1-r^2 \to du = -2r\,dr$
$\to -\frac{1}{2}du = r\,dr$; $r: 0 \to 1$ so $u: 1 \to 0$)

$= \int_1^0 \frac{-1}{2}\sqrt{u}\,du = -\frac{1}{2}\cdot \frac{u^{3/2}}{3/2}\Big|_1^0 = -\frac{1}{3}\left(0^{3/2} - 1^{3/2}\right)$

$= \left(-\frac{1}{3}\right)(-1) = 1/3$

4.) b.) $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$ (Let $u = \cos x \to$
$du = -\sin x\,dx \to -du = \sin x\,dx$;
$x: 2\pi \to 3\pi$ so $u: \cos 2\pi \to \cos 3\pi$ or
$u: 1 \to -1$)

$= \int_1^{-1} -3u^2\,du = -u^3\Big|_1^{-1} = -(-1)^3 - -(1)^3 = 2$

7.) b.) $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$ (Let $u = 4+r^2 \to$
$du = 2r\,dr \to$
$\frac{1}{2}du = r\,dr$; $r: 0 \to 1$ so $u: 4 \to 5$)

$= 5\left(\frac{1}{2}\right)\int_4^5 \frac{1}{u^2}\,du = \frac{5}{2}\int_4^5 u^{-2}\,du = \frac{5}{2}\cdot \frac{u^{-1}}{-1}\Big|_4^5$

$= -\frac{5}{2}\left(\frac{1}{5} - \frac{1}{4}\right) = -\frac{5}{2}\cdot \frac{-1}{20} = \frac{1}{8}$

11.) a.) $\int_0^{\frac{\pi}{6}} (1-\cos 3t)\sin 3t\,dt$ (Let $u = 1-\cos 3t \to$
$du = -(-\sin 3t \cdot 3)\,dt = 3\sin 3t\,dt \to$
$\frac{1}{3}du = \sin 3t\,dt$; $t: 0 \to \frac{\pi}{6}$ so
$u: (1-\cos 0) \to (1-\cos\frac{\pi}{2})$ or $u: 0 \to 1$)

$= \int_0^1 \frac{1}{3}\cdot u\,du = \frac{1}{3}\cdot \frac{u^2}{2}\Big|_0^1 = \frac{1}{6}(1-0) = 1/6$

16.) $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$ (Let $u = 1+\sqrt{y} \to du = \frac{1}{2}y^{-1/2}\,dy$
$= \frac{1}{2\sqrt{y}}\,dy$; $y: 1 \to 4$ so $u: 2 \to 3$)

$= \int_2^3 \frac{1}{u^2}\,du = \frac{-1}{u}\Big|_2^3 = \frac{-1}{3} - \frac{-1}{2} = \frac{1}{6}$

22.) $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

(Let $u = y^3+6y^2-12y+9 \rightarrow du = (3y^2+12y-12)\,dy$
$= 3(y^2+4y-4)\,dy \rightarrow \frac{1}{3}du = (y^2+4y-4)\,dy$;
$y: 0 \rightarrow 1$ so $u: 9 \rightarrow 4$)

$= \int_9^4 \frac{1}{3} u^{-1/2}\,du = \frac{1}{3}\cdot \frac{u^{1/2}}{1/2}\Big|_9^4 = \frac{2}{3}(2-3) = -\frac{2}{3}$

24.) $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$ (Let $u = 1+\frac{1}{t} \rightarrow$
$du = \frac{-1}{t^2}\,dt \rightarrow -du = t^{-2}dt$; $t: -1 \rightarrow -1/2$
so $u: 0 \rightarrow -1$)

$= -\int_0^{-1} \sin^2 u\,du$ (RECALL: $\cos 2\theta = 1-2\sin^2\theta$
$\rightarrow 2\sin^2\theta = 1-\cos 2\theta \rightarrow \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$.)

$= -\int_0^{-1} \frac{1}{2}(1-\cos 2\theta)\,d\theta = -\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{-1}$

$= -\frac{1}{2}\left(-1-\frac{1}{2}\sin(-2)\right) - -\frac{1}{2}\left(0-\frac{1}{2}\overset{0}{\cancel{\sin 0}}\right)$

$= \frac{1}{2} + \frac{1}{4}\cdot(-\sin 2) = \frac{1}{2} - \frac{\sin 2}{4}$

31.) $\int_2^4 \frac{dx}{x(\ln x)^2}$ (Let $u = \ln x \rightarrow du = \frac{1}{x}dx$;
$x: 2 \rightarrow 4$ so $u: \ln 2 \rightarrow \ln 4$)

$= \int_{\ln 2}^{\ln 4} \frac{1}{u^2}\,du = \frac{-1}{u}\Big|_{\ln 2}^{\ln 4} = \frac{-1}{\ln 4} - \frac{-1}{\ln 2}$

$= \frac{-1}{\ln 4} + \frac{2}{2\ln 2} = \frac{-1}{\ln 4} + \frac{2}{\ln 4} = \frac{1}{\ln 4}$

41.) $\int_0^1 \frac{4\,ds}{\sqrt{4-s^2}} = 4\int_0^1 \frac{ds}{\sqrt{4\left(1-\frac{s^2}{4}\right)}} = 4\cdot\left(\frac{1}{2}\right)\int_0^1 \frac{ds}{\sqrt{1-\left(\frac{s}{2}\right)^2}}$

(Let $u = \frac{s}{2} \rightarrow du = \frac{1}{2}ds$; $s: 0 \rightarrow 1$ so
$u: 0 \rightarrow \frac{1}{2}$)

$= 4\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,du = 4\arcsin u\Big|_0^{1/2}$

$= 4\arcsin\left(\frac{1}{2}\right) - 4\arcsin(0) = 4\left(\frac{\pi}{6}\right) = \frac{2\pi}{3}$

47.) (By symmetry) Area $= 2\int_0^2 x\sqrt{4-x^2}\,dx$

(Let $u = 4 - x^2 \rightarrow du = -2x\,dx \rightarrow \frac{-1}{2}du = x\,dx$;

$x: 0 \rightarrow 2$ so $u: 4 \rightarrow 0$)

$= 2\cdot\left(\frac{-1}{2}\right)\int_4^0 \sqrt{u}\,du = -\frac{u^{3/2}}{3/2}\Big|_4^0 = -\frac{2}{3}\left(0^{3/2} - 4^{3/2}\right)$

$= 0 + \frac{2}{3}(8) = 16/3$

53.) (By symmetry) Area $= 2\int_0^2 [2x^2 - (x^4 - 2x^2)]\,dx$

$= 2\int_0^2 (4x^2 - x^4)\,dx = 2\left(\frac{4}{3}x^3 - \frac{x^5}{5}\right)\Big|_0^2$

$= 2\left(\frac{32}{3} - \frac{32}{5}\right) = 2\cdot\left(\frac{64}{15}\right) = \frac{128}{15}$

54.) Area $= \int_0^1 (y^2 - y^3)\,dy = \left(\frac{1}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^1$

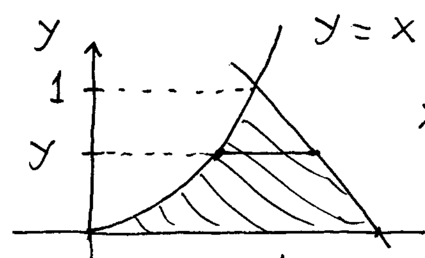
$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

55.) Area $= \int_0^1 [(12y^2 - 12y^3) - (2y^2 - 2y)]\,dy$

$= \int_0^1 [-12y^3 + 10y^2 + 2y]\,dy = \left(-3y^4 + \frac{10}{3}y^3 + y^2\right)\Big|_0^1$

$= -3 + \frac{10}{3} + 1 = 4/3$

58.)

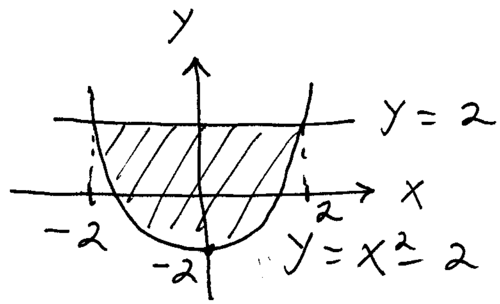


$y = x^2$ or $x = \sqrt{y}$

$x + y = 2$
or $x = 2 - y$

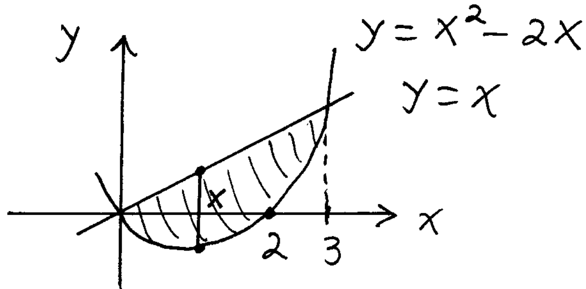
Area $= \int_0^1 [(2-y) - \sqrt{y}]\,dy$

$= \left(2y - \frac{y^2}{2} - \frac{y^{3/2}}{3/2}\right)\Big|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$

63.) 

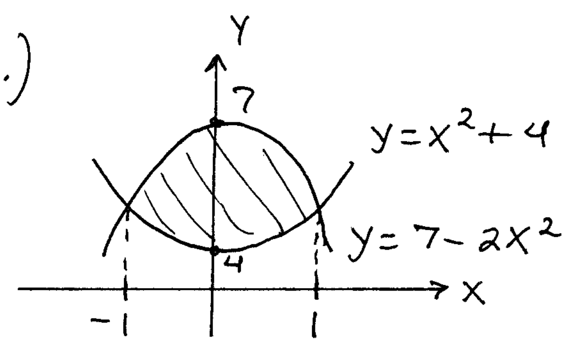
$x^2 - 2 = 2 \rightarrow x^2 = 4$
$\rightarrow x = \pm 2$

$$\text{Area} = \int_{-2}^{2} [2 - (x^2 - 2)]\, dx = \int_{-2}^{2} (4 - x^2)\, dx$$
$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$
$$= 16 - \frac{16}{3} = \frac{32}{3}$$

66.) 

$x^2 - 2x = x \rightarrow$
$x^2 - 3x = 0 \rightarrow$
$x(x-3) = 0 \rightarrow$
$x = 0$ or $x = 3$;

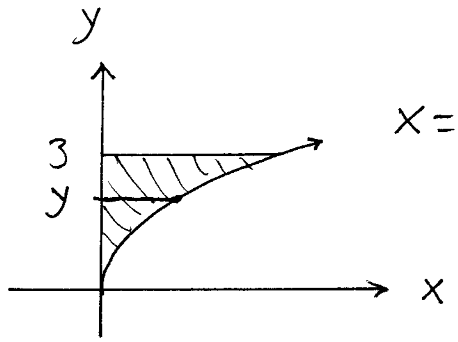
$$\text{Area} = \int_0^3 (x - (x^2 - 2x))\, dx = \int_0^3 (3x - x^2)\, dx$$
$$= \left(\frac{3}{2}x^2 - \frac{x^3}{3}\right)\Big|_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$$

68.) 

$x^2 + 4 = 7 - 2x^2 \rightarrow$
$3x^2 = 3 \rightarrow x^2 = 1 \rightarrow$
$x = \pm 1$

$$\text{Area} = \int_{-1}^{1} [(7 - 2x^2) - (x^2 + 4)]\, dx$$
$$= \int_{-1}^{1} [3 - 3x^2]\, dx = (3x - x^3)\Big|_{-1}^{1}$$
$$= (3-1) - (-3+1) = 2 - (-2) = 4$$

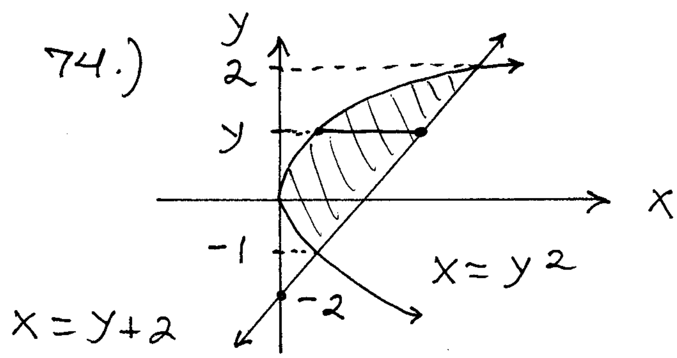
73.)



$x = 2y^2 \quad \text{Area} = \int_0^3 2y^2\,dy$

$= \frac{2}{3}y^3\Big|_0^3 = 18$

74.)



$y^2 = y + 2 \rightarrow$

$y^2 - y - 2 = 0 \rightarrow$

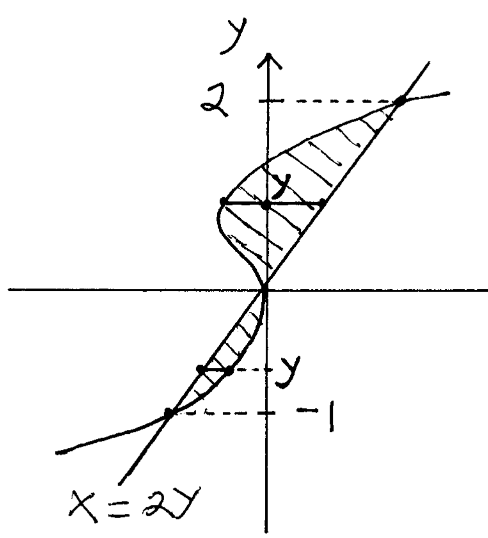
$(y-2)(y+1) = 0 \rightarrow$

$y = 2,\ y = -1$

$\text{Area} = \int_{-1}^{2} [(y+2) - y^2]\,dy = \left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right)\Big|_{-1}^{2}$

$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 8 - 3 - \frac{1}{2} = \frac{9}{2}$

80.)



$x = y^3 - y^2$

$y^3 - y^2 = 2y \rightarrow$

$y^3 - y^2 - 2y = 0 \rightarrow$

$y(y-2)(y+1) = 0 \rightarrow$

$y = 0,\ y = 2,\ \text{or}\ y = -1$

$\text{Area} = \int_{-1}^{0} [(y^3 - y^2) - 2y]\,dy$

$+ \int_0^2 [2y - (y^3 - y^2)]\,dy$

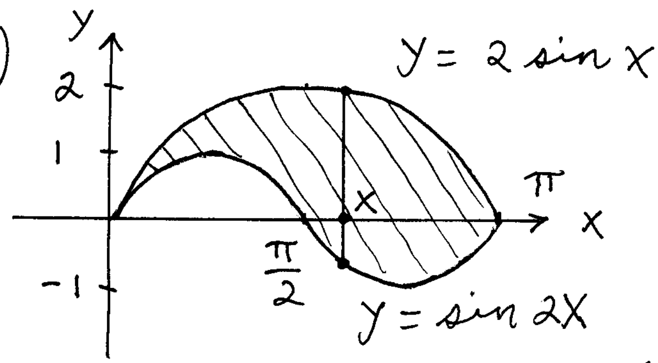
$= \left(\frac{1}{4}y^4 - \frac{1}{3}y^3 - y^2\right)\Big|_{-1}^{0}$

$+ \left(y^2 - \frac{1}{4}y^4 + \frac{1}{3}y^3\right)\Big|_0^2$

$$= (0) - \left(\frac{1}{4} + \frac{1}{3} - 1\right) + \left(4 - 4 + \frac{8}{3}\right) - (0)$$

$$= \frac{5}{12} + \frac{8}{3} = \frac{37}{12} .$$

85.)



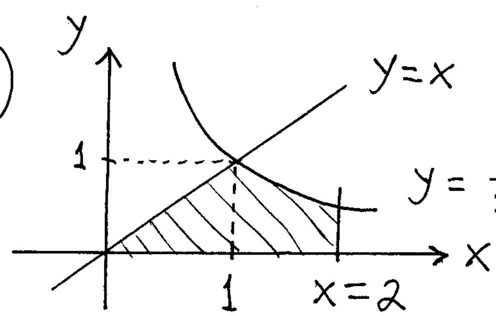
$$\text{Area} = \int_0^{\pi} (2\sin x - \sin 2x)\,dx$$

$$= \left(-2\cos x + \frac{1}{2}\cos 2x\right)\Big|_0^{\pi}$$

$$= \left(-2\cos\pi + \frac{1}{2}\cos 2\pi\right) - \left(-2\cos 0 + \frac{1}{2}\cos 0\right)$$

$$= +2 + \frac{1}{2} + 2 - \frac{1}{2} = 4$$

95.)

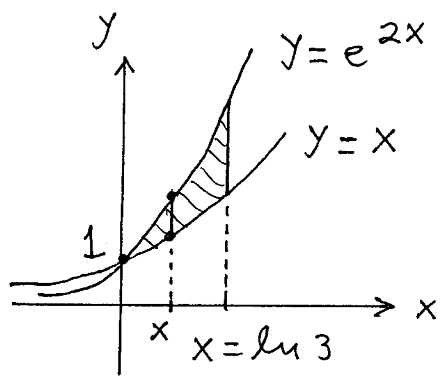


$$\text{Area} = \int_0^1 x\,dx + \int_1^2 \frac{1}{x^2}\,dx$$

$$= \frac{x^2}{2}\Big|_0^1 + \frac{-1}{x}\Big|_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left(\frac{-1}{2} - {}^-1\right) = 1$$

99.)



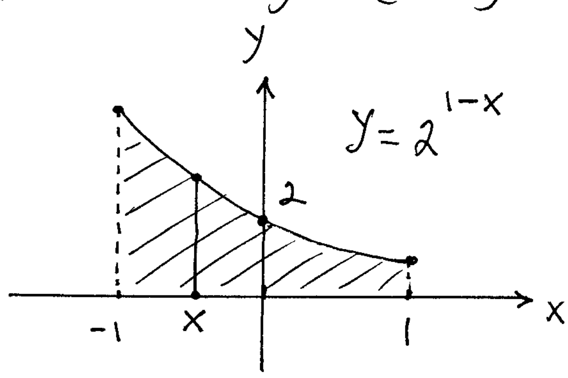
$$\text{Area} = \int_0^{\ln 3} (e^{2x} - e^x)\,dx$$

$$= \left(\frac{1}{2}e^{2x} - e^x\right)\Big|_0^{\ln 3}$$

$$= \left(\frac{1}{2}e^{2\ln 3} - e^{\ln 3}\right) - \left(\frac{1}{2}e^0 - e^0\right)$$

$$= \left(\frac{1}{2}e^{\ln 3^2} - 3\right) - \left(\frac{1}{2} - 1\right) = \frac{9}{2} - 3 + \frac{1}{2} = 2$$

102.)



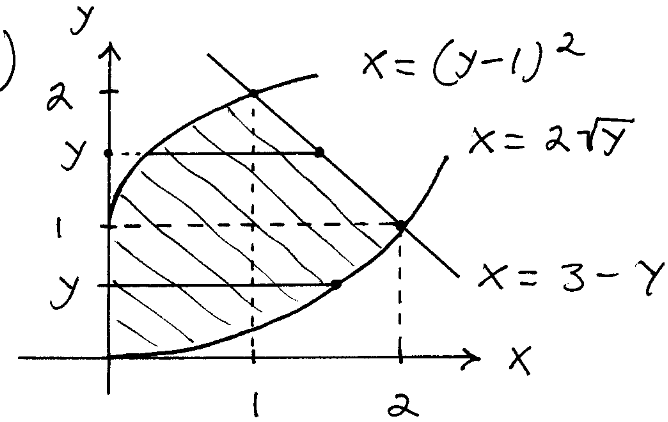
$$\text{Area} = \int_{-1}^1 2^{1-x}\,dx$$

$$= \frac{-1}{\ln 2}\, 2^{1-x}\Big|_{-1}^1$$

$$= \frac{-1}{\ln 2}\cdot 2^0 - \frac{-1}{\ln 2}\cdot 2^2$$

$$= \frac{-1}{\ln 2} + \frac{4}{\ln 2} = \frac{3}{\ln 2}$$

106.)



$$\text{Area} = \int_0^1 2\sqrt{y}\,dy + \int_1^2 \left[(3-y)-(y-1)^2\right] dy$$

115.) $I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)}$ (~~Let~~ $u = a - x \longrightarrow x = a-u$ and $du = -dx$; $x: 0 \to a$ ~~so~~ $u = a \to 0$)

$$= -\int_a^0 \frac{f(a-u)\,du}{f(a-u)+f(u)} = \int_0^a \frac{f(a-u)+f(u)-f(u)}{f(a-u)+f(u)}\,du$$

$$= \int_0^a \left(\frac{f(a-u)+f(u)}{f(a-u)+f(u)} - \frac{f(u)}{f(a-u)+f(u)} \right) du$$

$$= \int_0^a 1\,du - \int_0^a \frac{f(u)}{f(u)+f(a-u)}\,du$$

$$= u \Big|_0^a - I$$

$$= (a-0) - I$$

$$= a - I \quad \text{so that}$$

$$I = a - I \rightarrow 2I = a \rightarrow I = \frac{a}{2}$$