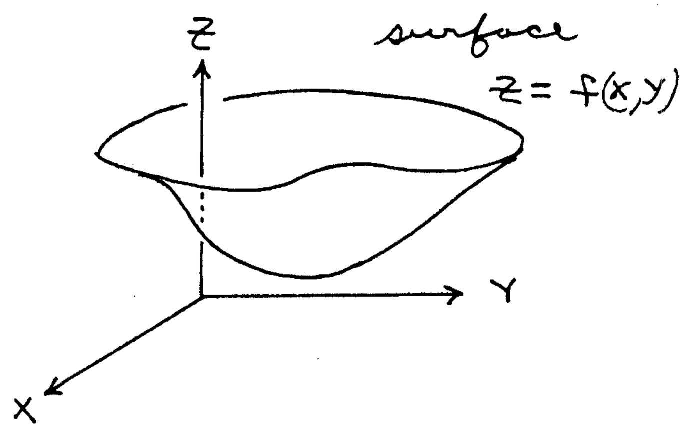
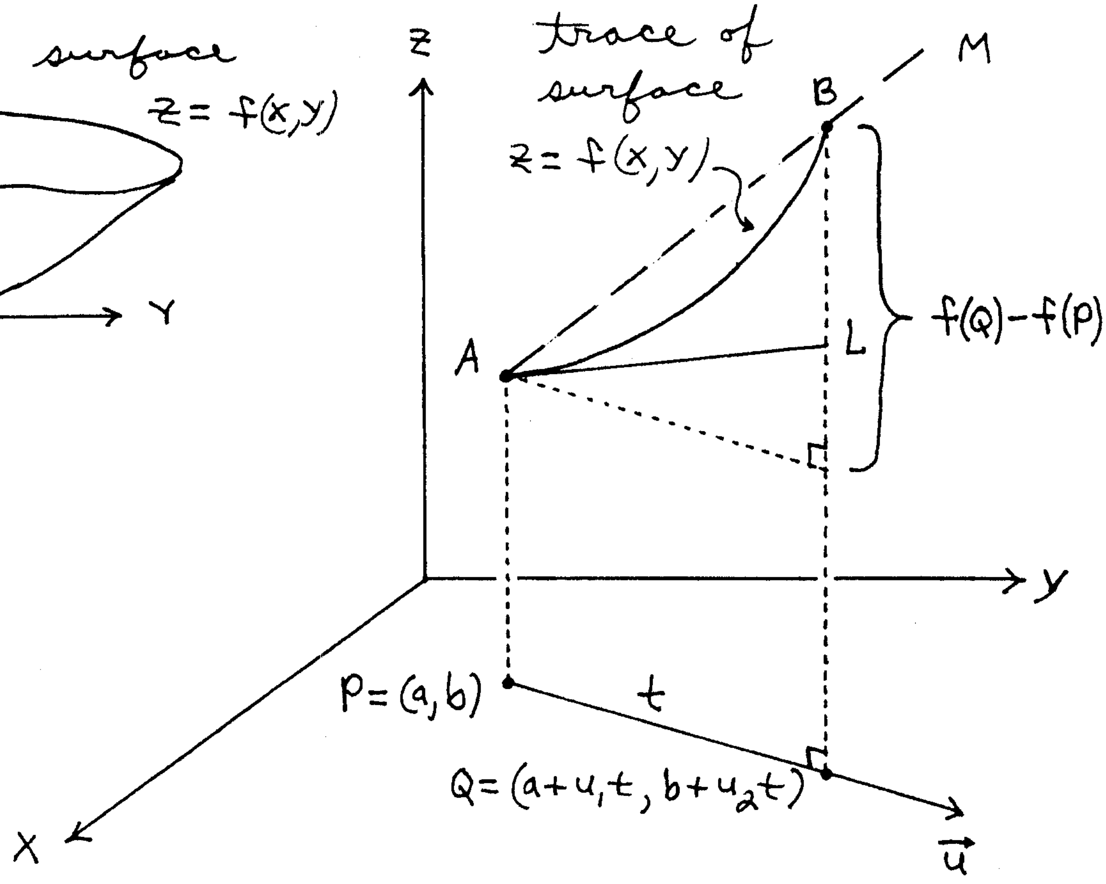
Math 21C

Vogler

Directional Derivatives and Gradient Vectors

Consider a surface given by the function $z = f(x,y)$, a point $P=(a,b)$, & a unit vector $\vec{u} = u_1\hat{i} + u_2\hat{j}$. We seek to define the derivative of f at a point P in the direction of $\vec{u}$.





Remarks:

1) Let point $Q = (a + u_1 t,\ b + u_2 t)$, where $t \geq 0$.

2) Vector $\overrightarrow{PQ} = \overrightarrow{(u_1 t, u_2 t)} = t\overrightarrow{(u_1, u_2)} = t\vec{u}$ points in the direction of $\vec{u}$ & has length t since $\vec{u}$ is a unit vector

3) Line M is secant line through points A and B.

4) Line L is line tangent to the trace of the surface $z = f(x,y)$ at the point $A = (a, b, f(a,b))$ in the direction of vector $\vec{u}$.

5) The <u>slope</u> of line M is

$$\frac{\text{rise}}{\text{run}} = \frac{f(Q) - f(P)}{t} = \frac{f(a+u_1 t, b+u_2 t) - f(a,b)}{t}$$

6) The <u>slope</u> of line L is

$$\lim_{t\to 0} \frac{f(a+u_1 t, b+u_2 t) - f(a,b)}{t}$$

<u>Defn</u> <u>Consider the function $z = f(x,y)$, point $P = (a,b)$, and unit vector $\vec{u} = u_1\hat{i} + u_2\hat{j}$. The derivative of f at point P in the direction $\vec{u}$ is</u>

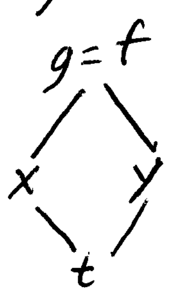
$$(\text{I}) \quad D_{\vec{u}} f(a,b) = \lim_{t\to 0} \frac{f(a+u_1 t, b+u_2 t) - f(a,b)}{t}$$

<u>Note</u>: $D_{\vec{u}} f(a,b)$ is the slope of tangent line L.

Now consider the function

$g(t) = f(a+u_1 t, b+u_2 t)$ or can be written as

$g(t) = f(x,y)$ with $x = a + u_1 t$ & $y = b + u_2 t$



By the chain rule it follows that

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \overrightarrow{\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)} \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

$$= \overrightarrow{\left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)} \cdot \overrightarrow{(u_1, u_2)} \quad \text{so that}$$

(II) $$g'(0) = \overrightarrow{\left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)} \cdot \vec{u}.$$

Note that $g(0) = f(a,b)$ so from (I) we obtain

$$D_{\vec{u}} f(a,b) = \lim_{t \to 0} \frac{g(t) - g(0)}{t - 0} = g'(0)$$

It now follows from (II) that

(III) $$\boxed{D_{\vec{u}} f(a,b) = \overrightarrow{\left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)} \cdot \vec{u}}$$

Defn: The gradient vector of $z = f(x,y)$ at point $P = (a,b)$ is given by

$$\vec{\nabla} f = \frac{\partial f}{\partial x}(a,b)\,\hat{i} + \frac{\partial f}{\partial y}(a,b)\,\hat{j} = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)$$

Notation: $D_{\vec{u}} f(a,b) = \vec{\nabla f} \cdot \vec{u}$

Moral: Taking the directional derivative of f in direction $\vec{u}$ (unit vector) is equivalent to finding gradient of f & dotting it with $\vec{u}$.

Note: 1) $D_{\vec{u}} f(a,b) = \vec{\nabla f}(a,b) \cdot \vec{u} = |\vec{\nabla} f(a,b)|\, \underbrace{|\vec{u}|}_{1} \cos\theta = |\vec{\nabla f}(a,b)| \cos\theta$

where θ is angle between $\vec{\nabla f}$ and $\vec{u}$.

2) The definitions for $D_{\vec{u}} f(a,b)$ and $\vec{\nabla f}$ generalize naturally to functions of more variables.

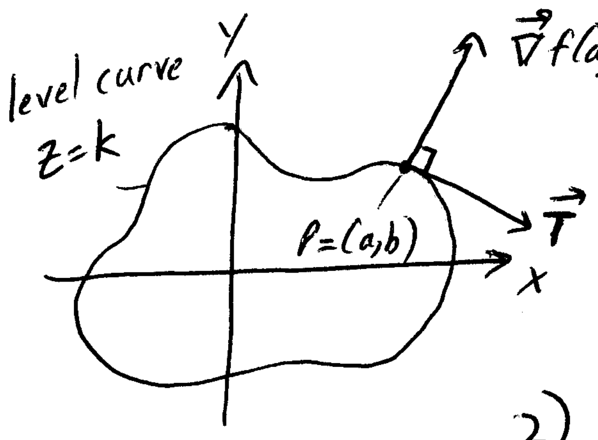
Facts Let $z = f(x,y)$ and let point $P=(a,b)$, with gradent vector $\vec{\nabla} f(a,b) = \overrightarrow{(f_x(a,b), f_y(a,b))}$

1) The max value of $D_{\vec{u}} f(a,b)$ is $|\vec{\nabla} f(a,b)|$ when vector $\vec{u}$ has same direction as $\vec{\nabla} f(a,b)$ (i.e. $\theta = 0°$). Also, f increases most rapidly in this direction.

2) The min value of $D_{\vec{u}} f(a,b)$ is $-|\vec{\nabla} f(a,b)|$ when vector $\vec{u}$ has opposite direction of $\vec{\nabla} f(a,b)$ (i.e $\theta = 180°$). Also, f decreases most rapidly in this direction.

Gradients and Level Curves

Let $z = f(x,y)$ and consider the level curve determined by point $P=(a,b)$ (i.e. $z = f(x,y) = k$ where $k = f(a,b)$ is constant)



Facts:

1) $\vec{\nabla} f(a,b)$ is orthogonal ($\perp$) to level curve of $z = f(x,y) = k$ determined by point (a,b).

2) A vector tangent to level curve $z = f(x,y) = k$ determined by point (a,b) is

$$\vec{T} = \overrightarrow{(f_y(a,b), -f_x(a,b))}$$