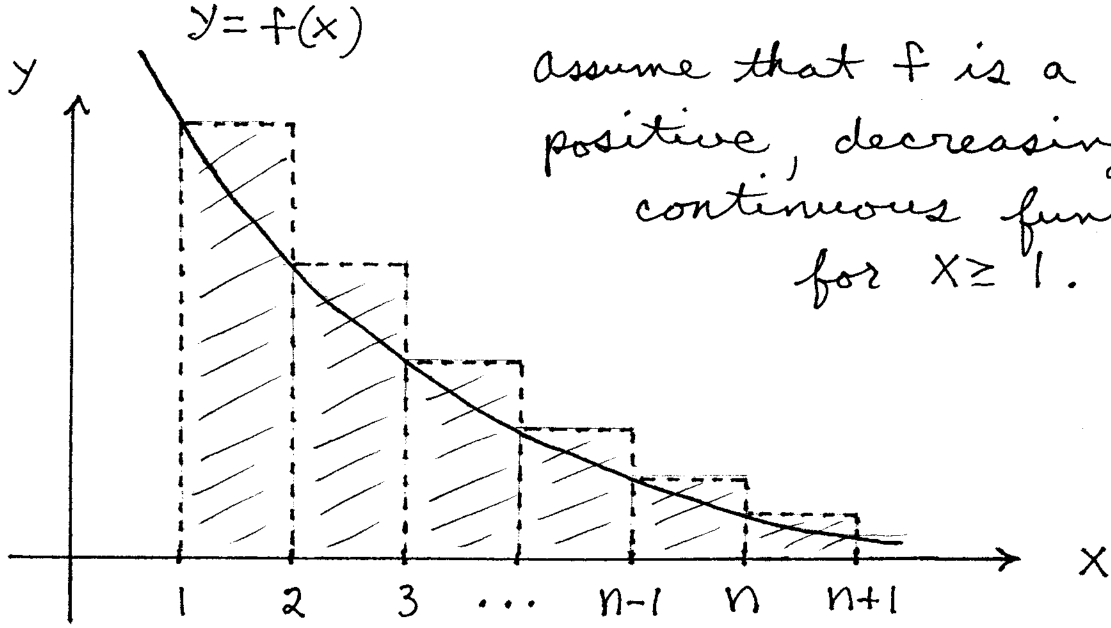
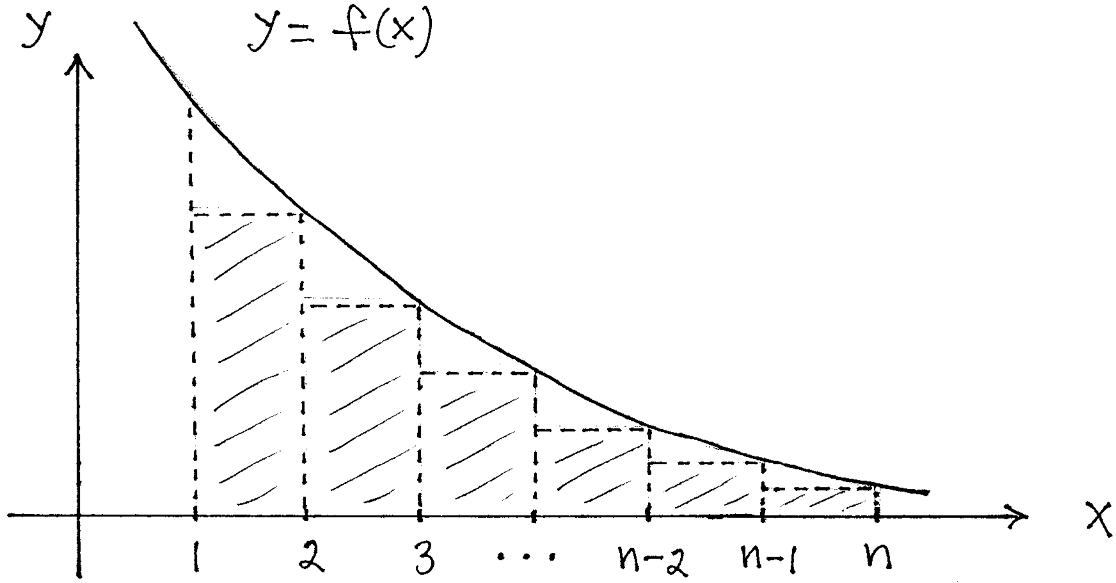
Infinite Series and Improper Integrals

Assume that f is a positive, decreasing, continuous function for $x \geq 1$.



$$\int_1^{n+1} f(x)\,dx < f(1) + f(2) + f(3) + \cdots + f(n) \;;$$



$$f(2) + f(3) + \cdots + f(n) < \int_1^{n} f(x)\,dx \implies$$

$$f(1) + f(2) + f(3) + \cdots + f(n) < f(1) + \int_1^{n} f(x)\,dx \;;$$

It follows that

$$(*) \quad \boxed{\int_1^{n+1} f(x)\,dx < f(1)+f(2)+\cdots+f(n) < f(1)+\int_1^n f(x)\,dx} \,.$$

Case 1: Assume that $\int_1^\infty f(x)\,dx = L$ (finite). Then

$$f(1)+f(2)+\cdots+f(n) < f(1)+\int_1^n f(x)\,dx \Rightarrow$$

$$\lim_{n\to\infty} (f(1)+f(2)+\cdots+f(n)) < \lim_{n\to\infty} \left(f(1)+\int_1^n f(x)\,dx\right) \Rightarrow$$

$$\sum_{n=1}^\infty f(n) < f(1)+\int_1^\infty f(x)\,dx = f(1)+L < \infty \,, \text{ i.e.,}$$

$$\sum_{n=1}^\infty f(n) \text{ converges .}$$

Case 2: Assume that $\int_1^\infty f(x)\,dx = \infty$. Then

$$\int_1^{n+1} f(x)\,dx < f(1)+f(2)+\cdots+f(n) \Rightarrow$$

$$\lim_{n\to\infty} \int_1^{n+1} f(x)\,dx \le \lim_{n\to\infty} (f(1)+f(2)+\cdots+f(n)) \Rightarrow$$

$$\int_1^\infty f(x)\,dx \le \sum_{n=1}^\infty f(n) \Rightarrow \sum_{n=1}^\infty f(n) = \infty \,, \text{ i.e.,}$$

$$\sum_{n=1}^\infty f(n) \text{ diverges .}$$

This verifies the following series test.

Integral Test: Assume that function f is positive, decreasing, and continuous for $x \ge 1$.

a.) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty f(n)$ converges.

b.) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty f(n)$ diverges.

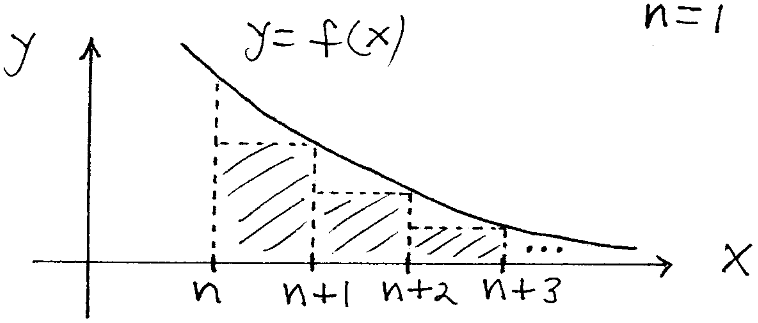
note that

$$\sum_{n=1}^{\infty} f(n) = \underbrace{f(1)+f(2)+\cdots+f(n)}_{S_n}+\underbrace{f(n+1)+f(n+2)+\cdots}_{R_n} \; ;$$

let $S_n = f(1)+f(2)+\cdots+f(n)$ be the nth partial sum, and let the infinite tail

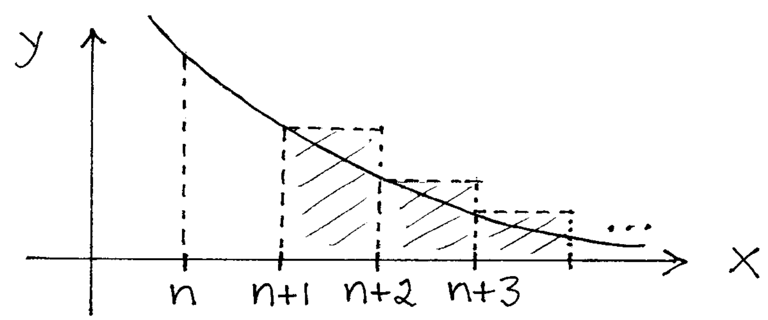
$$R_n = f(n+1)+f(n+2)+\cdots$$

be the error or remainder term for the infinite series $\sum_{n=1}^{\infty} f(n)$. Then



$$f(n+1)+f(n+2)+f(n+3)+\cdots < \int_n^{\infty} f(x)\,dx \, ,$$

and



$$\int_{n+1}^{\infty} f(x)\,dx < f(n+1)+f(n+2)+\cdots \, ,$$ so that

(*)(*) $$\boxed{\int_{n+1}^{\infty} f(x)\,dx < f(n+1)+f(n+2)+\cdots < \int_n^{\infty} f(x)\,dx}$$