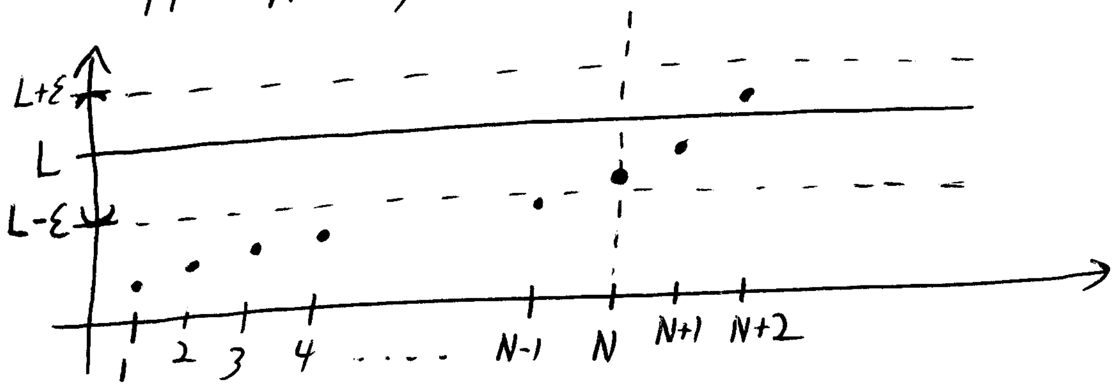
Math 21C
Vogler

Sequences

Defn A <u>sequence</u> $\{a_n\}$ is a function which assigns a real number to each natural number $n = 1, 2, 3, 4, \ldots$

Defn A sequence $\{a_n\}$ having a <u>limit</u> L (finite #), written $\lim_{n \to \infty} a_n = L$, means: For each number $\varepsilon > 0$, there exists an integer N, such that

if $n > N$, then $|a_n - L| < \varepsilon$.



Note: If an L exists, we say the sequence <u>converges</u>
Otherwise, we say it <u>diverges</u>

Thm Sandwich Theorem
Let $\{a_n\}$, $\{b_n\}$, & $\{c_n\}$ be sequences satisfying

$$b_n \le a_n \le c_n$$

If $\lim_{n\to\infty} b_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} a_n = L$

Example: Find the limit of the sequence $\left\{ \frac{3^n}{n!} \right\}$. Start by writing out the first six terms:

$$n: \quad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5 \qquad 6$$

$$\frac{3^n}{n!}: \quad \frac{3}{1}, \frac{3\cdot 3}{2\cdot 1}, \frac{3\cdot 3\cdot 3}{3\cdot 2\cdot 1}, \frac{3\cdot 3\cdot 3\cdot 3}{4\cdot 3\cdot 2\cdot 1}, \frac{3\cdot 3\cdot 3\cdot 3\cdot 3}{5\cdot 4\cdot 3\cdot 2\cdot 1}, \frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdot 3}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}, \dots$$

$$= 3, \frac{9}{2}, \frac{9}{2}, \left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \left(\frac{3}{5}\right)\left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \left(\frac{3}{6}\right)\left(\frac{3}{5}\right)\left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \dots$$

$$\leq 3, \frac{9}{2}, \left(\frac{3}{4}\right)^0\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^1\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^2\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^3\left(\frac{9}{2}\right), \dots$$

Thus, for $n \geq 3$ it follows that

$$0 \leq \frac{3^n}{n!} \leq \left(\frac{3}{4}\right)^{n-3}\left(\frac{9}{2}\right).$$

Since $\lim_{n\to\infty} \left(\frac{3}{4}\right)^{n-3}\left(\frac{9}{2}\right) = (0)\left(\frac{9}{2}\right) = 0$, it follows from the Sandwich Theorem that $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

Rule: Let x be any real number. Then

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$