Section 14.3

2.) $f(x,y) = x^2 - xy + y^2 \longrightarrow$

$$\frac{\partial f}{\partial x} = 2x - y \ , \quad \frac{\partial f}{\partial y} = -x + 2y$$

7.) $f(x,y) = (x^2+y^2)^{1/2} \longrightarrow$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot(2x) = \frac{x}{\sqrt{x^2+y^2}} \ ,$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

10.) $f(x,y) = \frac{x}{x^2+y^2} \longrightarrow$

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \ ,$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

13.) $f(x,y) = e^{x+y+1} \longrightarrow$

$$\frac{\partial f}{\partial x} = e^{x+y+1}\cdot(1) \ , \quad \frac{\partial f}{\partial y} = e^{x+y+1}\cdot(1)$$

16.) $f(x,y) = e^{xy}\ln y \longrightarrow$

$$\frac{\partial f}{\partial x} = e^{xy}\cdot(y)\cdot\ln y + e^{xy}\cdot(0) = ye^{xy}\ln y \ ,$$

$$\frac{\partial f}{\partial y} = e^{xy}\cdot\frac{1}{y} + xe^{xy}\cdot\ln y = e^{xy}\left(\frac{1}{y} + x\ln y\right)$$

21.) $f(x,y) = \int_x^y g(t)\,dt \longrightarrow f(x,y) = -\int_y^x g(t)\,dt$;

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(-\int_y^x g(t)\,dt\right) = -g(x) \quad (\text{by FTC 1}) \ ,$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int_x^y g(t)\,dt\right) = g(y) \quad (\text{by FTC I})$$

43.) $g(x,y) = x^2 y + \cos y + y \sin x \longrightarrow$

$g_x = 2xy + y\cos x$, $g_y = x^2 - \sin y + \sin x$,

$g_{xx} = 2y - y\sin x$, $g_{yy} = -\cos y$,

$g_{xy} = 2x + \cos x$, $g_{yx} = 2x + \cos x$

45.) $r(x,y) = \ln(x+y) \longrightarrow$

$r_x = \dfrac{1}{x+y}$, $r_y = \dfrac{1}{x+y}$,

$r_{xx} = \dfrac{-1}{(x+y)^2}$, $r_{yy} = \dfrac{-1}{(x+y)^2}$,

$r_{xy} = \dfrac{-1}{(x+y)^2}$, $r_{yx} = \dfrac{-1}{(x+y)^2}$

46.) $s(x,y) = \arctan\left(\frac{y}{x}\right) \longrightarrow$

$s_x = \dfrac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \dfrac{-y}{x^2} = \dfrac{-y}{x^2+y^2}$,

$s_y = \dfrac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \dfrac{1}{x} = \dfrac{1}{x+\frac{y^2}{x}} \cdot \dfrac{x}{x} = \dfrac{x}{x^2+y^2}$,

$s_{xx} = \dfrac{(x^2+y^2)(0)-(-y)(2x)}{(x^2+y^2)^2} = \dfrac{2xy}{(x^2+y^2)^2}$,

$s_{yy} = \dfrac{(x^2+y^2)(0)-x(2y)}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2}$,

$$S_{xy} = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

$$S_{yx} = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

54.) $f(x,y) = 4+2x-3y-xy^2$;

$$\frac{\partial f}{\partial x}(-2,1) = \lim_{h\to 0} \frac{f(-2+h,1)-f(-2,1)}{h}$$

$$= \lim_{h\to 0} \frac{[4+2(-2+h)-3(1)-(-2+h)(1)^2]-(-1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}+2h-\cancel{3}+\cancel{2}-h+\cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1 \;;$$

$$\frac{\partial f}{\partial y}(-2,1) = \lim_{h\to 0} \frac{f(-2,1+h)-f(-2,1)}{h}$$

$$= \lim_{h\to 0} \frac{[4+2(-2)-3(1+h)-(-2)(1+h)^2]-(-1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}-3-3h+2(1+2h+h^2)+1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{-3}-3h+\cancel{2}+4h+2h^2+\cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{2h^2+h}{h}$$

$$= \lim_{h\to 0} (2h+1) = 1.$$

65) assume $z = f(x,y)$ and $xy+z^3x-2yz=0$

$$\to \frac{\partial}{\partial x}(xy + z^3 x - 2yz) = \frac{\partial}{\partial x}(0)$$

$$\to y + \left(z^3 \cdot (1) + (3z^2 \cdot z_x) \cdot x\right) - 2y z_x = 0$$

$$\to y + z^3 + 3xz^2 \cdot z_x - 2y z_x = 0$$

$$\to (3xz^2 - 2y) z_x = -y - z^3$$

$$\to z_x = \frac{-y - z^3}{3xz^2 - 2y} \qquad (\text{Let } x=1,\ y=1,\ z=1)$$

$$\to z_x = \frac{-1-1}{3-2} = -2$$

75) Show $f(x,y) = e^{-2y}\cos 2x$ satisfies $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$:

$f_x = e^{-2y} \cdot -\sin 2x \cdot 2 = -2\sin 2x \cdot e^{-2y}$;

$f_y = -2e^{-2y} \cdot \cos 2x = -2\cos 2x \cdot e^{-2y}$;

$f_{xx} = -2 \cdot (2\cos 2x) \cdot e^{-2y} = -4\cos 2x \cdot e^{-2y}$;

$f_{yy} = -2\cos 2x \cdot (-2e^{-2y}) = 4\cos 2x \cdot e^{-2y}$;

then $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (-4\cos 2x \cdot e^{-2y}) + (4\cos 2x \cdot e^{-2y}) = 0$.

76) Show $\ln(x^2+y^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2)$ satisfies $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$:

$f_x = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2x = \frac{x}{x^2+y^2}$,

$f_y = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2y = \frac{y}{x^2+y^2}$,

$$f_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2},$$

$$f_{yy} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}; \text{ then}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$= \frac{\cancel{y^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0.$$

81) Show $w = \sin(x+ct)$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = c^2 \cdot \frac{\partial^2 w}{\partial x^2}:$$

$$\frac{\partial w}{\partial t} = \cos(x+ct) \cdot c = c \cdot \cos(x+ct),$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) \cdot (1) = \cos(x+ct),$$

$$\frac{\partial^2 w}{\partial t^2} = c \cdot -\sin(x+ct) \cdot c = -c^2 \sin(x+ct),$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) \cdot (1) = -\sin(x+ct);$$

then $$\frac{\partial^2 w}{\partial t^2} = -c^2 \sin(x+ct)$$

$$= c^2 (-\sin(x+ct))$$

$$= c^2 \cdot \frac{\partial^2 w}{\partial x^2}.$$

84) Show $w = \ln(2x + 2ct)$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = c^2 \cdot \frac{\partial^2 w}{\partial x^2} \quad :$$

$$\frac{\partial w}{\partial t} = \frac{1}{2x+2ct} \cdot (2c) = \frac{\cancel{2}c}{\cancel{2}(x+ct)} = \frac{c}{x+ct} \, ,$$

$$\frac{\partial w}{\partial x} = \frac{1}{2x+2ct} \cdot (2) = \frac{\cancel{2}}{\cancel{2}(x+ct)} = \frac{1}{x+ct} \, ,$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c \cdot (x+ct)^{-1}\right) = -c(x+ct)^{-2} \cdot (c)$$

$$= \frac{-c^2}{(x+ct)^2} \, ,$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}(x+ct)^{-1} = -(x+ct)^{-2} \cdot (1) = \frac{-1}{(x+ct)^2} \, ;$$

then

$$\frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x+ct)^2}$$

$$= c^2 \cdot \frac{-1}{(x+ct)^2}$$

$$= c^2 \cdot \frac{\partial^2 w}{\partial x^2}$$