Section 10.4

18) $\dfrac{3}{n+\sqrt{n}} \geq \dfrac{3}{n+n} = \dfrac{3}{2n} = \dfrac{3}{2}\cdot\dfrac{1}{n}$ and

$\displaystyle\sum_{n=1}^{\infty} \frac{3}{2}\cdot\frac{1}{n} = \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges by p-series test since $p = 1 \leq 1$; so $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n+\sqrt{n}}$ diverges by comparison test

19) $0 \leq \dfrac{\sin^2 n}{2^n} \leq \dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$ and

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges by geometric series test; so $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n}{2^n}$ converges by comparison test

22) $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^2\cdot\sqrt{n}} = \sum_{n=1}^{\infty}\left(\frac{n}{n^2\sqrt{n}} + \frac{1}{n^2\sqrt{n}}\right)$

$\displaystyle = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}} + \sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$; each of these series converges by p-series test since $p = 3/2 > 1$ and $p = \frac{5}{2} > 1$; thus, $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^2\sqrt{n}}$ converges since it is the sum of convergent series

26) $0 \leq \dfrac{1}{\sqrt{n^3+2}} \leq \dfrac{1}{\sqrt{n^3+0}} = \dfrac{1}{n^{3/2}}$ and

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges by p-series test since $p = \frac{3}{2} > 1$; so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+2}}$ converges by comparison test

27) $\lim_{n\to\infty} \dfrac{\frac{1}{\ln(\ln n)}}{\frac{1}{n}} = \lim_{n\to\infty} \dfrac{n}{\ln(\ln n)}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{1}{\frac{1}{\ln n}\cdot\frac{1}{n}} = \lim_{n\to\infty} n \ln n = \infty$;

since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges by p-series test $(p = 1 \geq 1)$, then $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges by limit comparison test

28) $\lim_{n\to\infty} \dfrac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \dfrac{(\ln n)^2}{n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{2\cdot\ln n\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \dfrac{2\ln n}{n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{2\cdot\frac{1}{n}}{1} = 0$; since $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges by p-series test $(p = 2 > 1)$, then $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges by limit comparison test

30) $\lim_{n\to\infty} \dfrac{\frac{(\ln n)^2}{n^{3/2}}}{\frac{1}{n^{5/4}}} = \lim_{n\to\infty} \dfrac{(\ln n)^2}{n^{1/4}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{2\cdot\ln n\cdot\frac{1}{n}}{\frac{1}{4}n^{-3/4}} = \lim_{n\to\infty} 8\cdot\dfrac{\ln n}{n^{1/4}}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} 8\cdot\dfrac{1/n}{1/4\, n^{-3/4}} = \lim_{n\to\infty} 32\cdot\dfrac{1}{n^{1/4}} = 0$;

since $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges by p-series test ($p = \frac{5}{4} > 1$), then $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}}$ converges by the limit comparison test

31) $\lim_{n\to\infty} \dfrac{\frac{1}{1+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \dfrac{n}{1+\ln n}$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \dfrac{1}{1/n} = \lim_{n\to\infty} n = \infty$;

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test ($p = 1 \le 1$), then $\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$ diverges by limit comparison test

33) $\lim_{n\to\infty} \dfrac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \dfrac{n}{\sqrt{n^2-1}}$

$= \lim_{n\to\infty} \dfrac{\sqrt{n^2}}{\sqrt{n^2-1}} = \lim_{n\to\infty} \sqrt{\dfrac{n^2}{n^2-1}}$

$= \lim_{n\to\infty} \sqrt{\dfrac{1}{1-\left(\frac{1}{n^2}\right)}} = \sqrt{\dfrac{1}{1-0}} = 1$;

since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by p-series test ($p = 2 > 1$), then $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges by limit comparison test

34) $\lim_{n\to\infty} \dfrac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \dfrac{n^2}{n^2+1}$

$= \lim_{n\to\infty} \dfrac{1}{1+\frac{1}{n^2}} = \dfrac{1}{1+0} = 1$; since $\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges by p-series test $(p = \frac{3}{2} > 1)$, then $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}}{n^2+1}$ converges by limit comparison test

36) $\lim_{n\to\infty} \dfrac{\frac{n+2^n}{n^2 2^n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \dfrac{n+2^n}{2^n}$

$= \lim_{n\to\infty} \left(\dfrac{n}{2^n}+1\right) \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{n\to\infty} \left(\dfrac{1}{2^n \ln 2}+1\right)$ 0

$= 0+1 = 1$; since $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges by p-series test $(p = 2 > 1)$, then $\sum_{n=1}^{\infty} \dfrac{n+2^n}{n^2 2^n}$ converges by limit comparison test

37) $\lim_{n\to\infty} \dfrac{\frac{1}{3^{n-1}+1}}{\left(\frac{1}{3}\right)^n} = \lim_{n\to\infty} \dfrac{3^n}{3^{n-1}+1} \cdot \dfrac{\frac{1}{3^n}}{\frac{1}{3^n}}$

$= \lim_{n\to\infty} \dfrac{1}{\frac{1}{3}+\frac{1}{3^n}} = \dfrac{1}{\frac{1}{3}+0} = 3$; since

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by geometric series test ($r = \frac{1}{3}$, $-1 < r < 1$), then $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}+1}$ converges by limit comparison test

45) $$\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{n \to \infty} \frac{\cos\left(\frac{1}{n}\right) \cdot \cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}}$$

$= \cos 0 = 1$; since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test ($p = 1 \le 1$), then $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges by limit comparison test

47) $\lim_{n\to\infty} \dfrac{\frac{\arctan n}{n^{1.1}}}{\frac{1}{n^{1.1}}} = \lim_{n\to\infty} \arctan n$

$= \arctan(\infty) = \frac{\pi}{2}$; since $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges by p-series test ($p=1.1>1$), then $\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.1}}$ converges by limit comparison test

52) $\lim_{n\to\infty} \dfrac{\frac{n^{1/n}}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} n^{1/n} = \text{"}\infty^0\text{"}$ (indeterminate)

$= \lim_{n\to\infty} e^{\ln n^{1/n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n}$

$= e^{\lim_{n\to\infty} \frac{\ln n}{n}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} e^{\lim_{n\to\infty} \frac{\frac{1}{n}}{1}} = e^0 = 1$;

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series ($p=2>1$), then $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2}$ converges by limit comparison test

54) $\sum_{n=1}^{\infty} \frac{1}{1^2+2^2+3^2+\cdots+n^2}$

$= \sum_{n=1}^{\infty} \dfrac{1}{\frac{n(n+1)(2n+1)}{6}} = \sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)}$;

$\lim_{n\to\infty} \dfrac{\frac{6}{n(n+1)(2n+1)}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{6n^3}{n(n+1)(2n+1)}$

$= \lim_{n\to\infty} 6 \cdot \frac{\cancel{n}}{\cancel{n}} \cdot \frac{n}{n+1} \cdot \frac{n}{2n+1}$

$= \lim_{n\to\infty} 6 \cdot \frac{1}{1+\frac{1}{n}} \cdot \frac{1}{2+\frac{1}{n}} = 6 \cdot \frac{1}{1+0} \cdot \frac{1}{2+0}$

$= \frac{6}{2} = 3$; since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series test $(p = 3 > 1)$, then $\sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)}$ converges by limit comparison test

56) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

<u>Proof</u>: $0 \leq \frac{a_n}{n} \leq a_n$ and $\sum_{n=1}^{\infty} a_n$ converges, so $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges by comparison test.

58) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n^2$ converges.

<u>Proof</u>: Since $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ (by contrapositive of nth-term test). Since

$\lim_{n\to\infty} a_n = 0$ there is some integer N so that $0 \le a_n < 1$ for all $n \ge N$. Thus

$0 \le a_n^2 \le a_n$ for all $n \ge N$

and $\sum_{n=N}^{\infty} a_n$ converges, so

$\sum_{n=N}^{\infty} a_n^2$ converges by comparison test. It follows that

$$\sum_{n=1}^{\infty} a_n^2 = \underbrace{\sum_{n=1}^{N-1} a_n^2}_{\text{finite } \#} + \underbrace{\sum_{n=N}^{\infty} a_n^2}_{\text{finite } \#}$$

converges.