18) \[ \frac{3}{n+n} \geq \frac{3}{2n} = \frac{3}{2} \cdot \frac{1}{n} \]
and
\[ \sum_{n=1}^{\infty} \frac{3}{2} \cdot \frac{1}{n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n} \] diverges by
\( p \)-series test since \( p = 1 \leq 1 \); so
\[ \sum_{n=1}^{\infty} \frac{3}{n+n} \]
diverges by comparison test

19) \[ 0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n} = \left( \frac{1}{2} \right)^n \]
and
\[ \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \] converges by geometric series test; so \[ \sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n} \] converges by comparison test

22) \[ \frac{n+1}{n^2 \sqrt{n}} = \frac{1}{\sqrt{n}} \left( \frac{n}{n^2 \sqrt{n}} + \frac{1}{n^2 \sqrt{n}} \right) \]
\[ = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \] each of these
series converges by \( p \)-series test
since \( p = \frac{3}{2} > 1 \) and \( p = \frac{5}{2} > 1 \); thus,
\[ \sum_{n=1}^{\infty} \frac{n+1}{n^2 \sqrt{n}} \] converges since it is
the sum of convergent series

26) \[ 0 \leq \frac{1}{\sqrt{n^3+2}} \leq \frac{1}{\sqrt{n^3+0}} = \frac{1}{n^{3/2}} \]
and
\[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \]
converges by \( p \)-series test
since \( p = \frac{3}{2} > 1 \); so
\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2}} \]
converges by comparison test
27) \( \lim_{n \to \infty} \frac{1}{\ln(n)} = \lim_{n \to \infty} \frac{n}{n \ln(n)} \)

\[ \text{since } \frac{1}{n} \text{ diverges by p-series test } \left( p = 1 \geq 1 \right), \text{ then } \sum_{n=3}^{\infty} \frac{1}{\ln(n)} \]

diverges by limit comparison test

28) \( \lim_{n \to \infty} \frac{(\ln(n))^2}{n^3} = \lim_{n \to \infty} \frac{(\ln(n))^2}{n} \)

\[ \text{since } \frac{1}{n^2} \text{ converges by p-series test } \left( p = 2 \geq 1 \right), \text{ then } \sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n^3} \text{ converges by limit comparison test} \]

30) \( \lim_{n \to \infty} \frac{(\ln(n))^2}{n^{3/4}} = \lim_{n \to \infty} \frac{(\ln(n))^2}{\frac{1}{\sqrt[4]{n}}} \)

\[ \text{since } \frac{\ln(n)}{n^{3/4}} = 0 \]
since \( \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} \) converges by \( \rho \)-series test \((\rho = \frac{5}{4} > 1)\), then \( \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{3/2}} \) converges by the limit comparison test.

\[
31) \quad \lim_{n \to \infty} \frac{1}{1 + \ln n} = \lim_{n \to \infty} \frac{n}{1 + \ln n} \\
````
````
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
```
34) \( \lim_{n \to \infty} \frac{\sqrt{n}}{n^2 + 1} = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} \)

\[
= \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}} = \frac{1}{1+0} = 1 ; \text{ since } \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges by p-series test } (\rho = \frac{3}{2} > 1), \text{ then } \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1} \text{ converges by limit comparison test }
\]

36) \( \lim_{n \to \infty} \frac{n+2^n}{n^2 2^n} = \lim_{n \to \infty} \frac{n+2^n}{2^n} = 0 \)

\[
= \lim_{n \to \infty} \left( \frac{n}{2^n} + 1 \right) = \lim_{n \to \infty} \left( \frac{2^n}{2^n} + 1 \right) = 0 + 1 = 1 ; \text{ since } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges by p-series test } (\rho = 2 > 1), \text{ then } \sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} \text{ converges by limit comparison test }
\]

37) \( \lim_{n \to \infty} \frac{3^{n-1} + 1}{\left( \frac{1}{3} \right)^n} = \lim_{n \to \infty} \frac{3^n}{3^{n-1} + 1} \cdot \frac{\frac{1}{3^n}}{\frac{1}{3^n}} \)

\[
= \lim_{n \to \infty} \frac{1}{\frac{1}{3} + \frac{1}{3^n}} = \frac{1}{\frac{1}{3} + 0} = 3 ; \text{ since }
\]
\[ \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n \] converges by geometric series test \((r = \frac{1}{3}, -1 < r < 1)\),
then \[ \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} \] converges by limit comparison test.

45) \[ \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} \overset{0}{=} \lim_{n \to \infty} \frac{\cos \left( \frac{1}{n} \right) \cdot -\frac{1}{n^2}}{\frac{1}{n^2}} \]
\[ = \cos 0 = 1 \] since \[ \sum_{n=1}^{\infty} \frac{1}{n} \] diverges by \(p\)-series test \((p = 1 \leq 1)\), then \[ \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \] diverges by limit comparison test.
47) \( \lim_{n \to \infty} \frac{\arctan n}{n^{1.1}} = \lim_{n \to \infty} \arctan n \frac{\frac{1}{n^{1.1}}}{1} \)

= \arctan (\infty) = \frac{\pi}{2} \), since \( \sum_{n=1}^{\infty} \frac{1}{n^{1.1}} \)

converges by p-series test \((p = 1.1 > 1)\), then \( \sum_{n=1}^{\infty} \frac{1}{n^{1.1}} \) converges by

limit comparison test.

52) \( \lim_{n \to \infty} \frac{n^{1/n}}{n^{1/n}} = \lim_{n \to \infty} n^{1/n} = \infty \) (indeterminate)

= \lim_{n \to \infty} e^{\ln n^{1/n}} = \lim_{n \to \infty} e^{\frac{1}{n} \ln n}

= \lim_{n \to \infty} \frac{\ln n}{n} = \infty \lim_{n \to \infty} \frac{1}{n} = 0 \),

since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges by p-series \((p = 2 > 1)\), then \( \sum_{n=1}^{\infty} \frac{n^{1/n}}{n^2} \) converges by limit comparison test.

54) \( \sum_{n=1}^{\infty} \frac{1}{1^2 + 2^2 + 3^2 + \ldots + n^2} \)

= \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)(2n+1)} = \sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)} \)

= \( \lim_{n \to \infty} \frac{6}{n(n+1)(2n+1)} = \lim_{n \to \infty} \frac{6n^3}{n(n+1)(2n+1)} \)

= \( \lim_{n \to \infty} \frac{6n^3}{n^3} = \lim_{n \to \infty} \frac{6}{(n+1)(2n+1)} \)
\[ \lim_{n \to \infty} \frac{6 \cdot \frac{n}{n+1} \cdot \frac{n}{2n+1}}{1 + \frac{1}{n}} = \lim_{n \to \infty} \frac{6}{1 + \frac{1}{n}} = 6 \cdot \frac{1}{1+0} = 6 \]

\[ \frac{6}{2} = 3; \text{ since } \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converged by p-series test (} p = 3 > 1 \text{)} , \]

by p-series test \( p = 3 > 1 \), then \( \sum_{n=1}^{\infty} \frac{6}{n(n+1)(2n+1)} \) converges by limit comparison test.

56) If \( \sum_{n=1}^{\infty} \frac{a_n}{n} \) converges and \( a_n \geq 0 \),

then \( \sum_{n=1}^{\infty} \frac{a_n}{n} \) converges.

**Proof:** \( 0 \leq \frac{a_n}{n} \leq a_n \) and

\( \sum_{n=1}^{\infty} a_n \) converges, so \( \sum_{n=1}^{\infty} \frac{a_n}{n} \) converges by comparison test.

58) If \( \sum_{n=1}^{\infty} a_n \) converges and \( a_n \geq 0 \),

then \( \sum_{n=1}^{\infty} a_n^2 \) converges.

**Proof:** Since \( \sum_{n=1}^{\infty} a_n \) converges,

then \( \lim_{n \to \infty} a_n = 0 \) (by contrapositive of n-th term test).

Since
\[ \lim_{n \to \infty} a_n = 0 \] there is some integer \( N \) so that \( 0 \leq a_n < 1 \) for all \( n \geq N \). Thus
\[ 0 \leq a_n^2 \leq a_n \text{ for all } n \geq N \]
and \( \sum_{n=N}^{\infty} a_n \) converges, so
\[ \sum_{n=N}^{\infty} a_n^2 \text{ converges by comparison test.} \]
It follows that
\[ \sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{N-1} a_n^2 + \sum_{n=N}^{\infty} a_n^2 \]
\[ \text{finite} \quad \text{finite} \]
converges.