Section 10.8

Taylor polynomial of degree n:

$$P_n(x;a) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n,$$ where

$$a_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n = 0, 1, 2, 3, \ldots$$

3) $f(x) = \ln x$, $a = 1$:

$f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, so

$a_0 = \frac{f(1)}{0!} = \frac{\ln 1}{1} = \frac{0}{1} = 0$,

$a_1 = \frac{f'(1)}{1!} = \frac{1}{1} = 1$, $a_2 = \frac{f''(1)}{2!} = \frac{-1}{2}$,

$a_3 = \frac{f'''(1)}{3!} = \frac{2}{6} = \frac{1}{3}$; then

$P_0(x;1) = a_0 = 0$,

$P_1(x;1) = a_0 + a_1(x-1) = 1\cdot(x-1) = x-1$,

$P_2(x;1) = a_0 + a_1(x-1) + a_2(x-1)^2$

$= (x-1) + \frac{-1}{2}(x-1)^2$,

$P_3(x;1) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$

$= (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

6) $f(x) = \frac{1}{x+2} = (x+2)^{-1}$, $a = 0$:

$f'(x) = -(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$,

$f'''(x) = -6(x+2)^{-4}$, so

$a_0 = \frac{f(0)}{0!} = \frac{1/2}{1} = 1/2$, $a_1 = \frac{f'(0)}{1!} = \frac{-1/4}{1} = -1/4$,

$a_2 = \frac{f''(0)}{2!} = \frac{2 \cdot 1/8}{2} = \frac{1}{8}$,

$a_3 = \frac{f'''(0)}{3!} = \frac{-6 \cdot 1/16}{6} = \frac{-1}{16}$; then

$P_0(x;0) = a_0 = 1/2$,

$P_1(x;0) = a_0 + a_1 x = \frac{1}{2} - \frac{1}{4}x$,

$P_2(x;0) = a_0 + a_1 x + a_2 x^2$

$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2$,

$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3$

7) $f(x) = \cos x$, $a = \frac{\pi}{4}$:

$f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$,

so $a_0 = \frac{f(\frac{\pi}{4})}{0!} = \frac{\cos\frac{\pi}{4}}{1} = \frac{\sqrt{2}}{2}$,

$a_1 = \frac{f'(\frac{\pi}{4})}{1!} = \frac{-\sin\frac{\pi}{4}}{1} = \frac{-\sqrt{2}}{2}$,

$a_2 = \frac{f''(\frac{\pi}{4})}{2!} = \frac{-\cos\frac{\pi}{4}}{2} = \frac{-\frac{\sqrt{2}}{2}}{2} = \frac{-\sqrt{2}}{4}$,

$a_3 = \frac{f'''(\frac{\pi}{4})}{3!} = \frac{\sin\frac{\pi}{4}}{6} = \frac{\frac{\sqrt{2}}{2}}{6} = \frac{\sqrt{2}}{12}$; then

$P_0(x;\frac{\pi}{4}) = a_0 = \frac{\sqrt{2}}{2}$,

$P_1(x;\frac{\pi}{4}) = a_0 + a_1(x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \frac{\pi}{4})$,

$P_2(x;\frac{\pi}{4}) = a_0 + a_1(x - \frac{\pi}{4}) + a_2(x - \frac{\pi}{4})^2$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 ,$$

$$P_3\left(x; \frac{\pi}{4}\right) = a_0 + a_1\left(x-\frac{\pi}{4}\right) + a_2\left(x-\frac{\pi}{4}\right)^2 + a_3\left(x-\frac{\pi}{4}\right)^3$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3$$

10) $f(x) = \sqrt{x+4}$, $a = 0$:

$f'(x) = \frac{1}{2}(x+4)^{-1/2}$, $f''(x) = \frac{-1}{4}(x+4)^{-3/2}$,

$f'''(x) = \frac{3}{8}(x+4)^{-5/2}$; so

$a_0 = \frac{f(0)}{0!} = \frac{\sqrt{4}}{1} = 2$, $a_1 = \frac{f'(0)}{1!} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{1} = \frac{1}{4}$,

$a_2 = \frac{f''(0)}{2!} = \frac{\frac{-1}{4}\cdot\frac{1}{8}}{2} = \frac{-1}{64}$,

$a_3 = \frac{f'''(0)}{3!} = \frac{3/8 \cdot 1/32}{6} = \frac{1}{512}$; then

$P_0(x;0) = a_0 = 2$,

$P_1(x;0) = a_0 + a_1 x = 2 + \frac{1}{4}x$,

$P_2(x;0) = a_0 + a_1 x + a_2 x^2 = 2 + \frac{1}{4}x - \frac{1}{64}x^2$,

$P_3(x;0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3$

11) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so

$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$

$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^n}{n!}$

13) $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots = \sum_{n=0}^{\infty} x^n$, so

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1+(-x)+(-x)^2+(-x)^3+\cdots$$

$$= 1-x+x^2-x^3+x^4-\cdots = \sum_{n=0}^{\infty}(-1)^n x^n$$

15) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$,

so $\sin 3x = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \cdots$

$$= 3x - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7 + \cdots = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{3^{2n+1}}{(2n+1)!}x^{2n+1}$$

25) $f(x) = x^4 + x^2 + 1$, $a = -2$:

$f'(x) = 4x^3 + 2x$, $f''(x) = 12x^2 + 2$,

$f'''(x) = 24x$, $f^{(4)}(x) = 24$, $f^{(5)}(x) = 0$,

$f^{(6)}(x) = f^{(7)}(x) = f^{(8)}(x) = \cdots = 0$; ~~so~~

$a_0 = \frac{f(-2)}{0!} = \frac{21}{1} = 21$, $a_1 = \frac{f'(-2)}{1!} = \frac{-36}{1} = -36$,

$a_2 = \frac{f''(-2)}{2!} = \frac{50}{2} = 25$, $a_3 = \frac{f'''(-2)}{3!} = \frac{-48}{6} = -8$,

$a_4 = \frac{f^{(4)}(-2)}{4!} = \frac{24}{24} = 1$, $a_5 = a_6 = a_7 = \cdots = 0$;

then $x^4 + x^2 + 1 = a_0 + a_1(x-(-2)) + a_2(x-(-2))^2 + \cdots$

$= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4 + 0(x+2)^5 + \cdots$

$= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4$

27) $f(x) = \frac{1}{x^2}$, $a = 1$:

$f'(x) = -2x^{-3}$, $f''(x) = 3 \cdot 2 x^{-4} = 3!\, x^{-4}$,

$f'''(x) = -4 \cdot 3!\, x^{-5} = -4!\, x^{-5}$,

$f^{(4)}(x) = 5 \cdot 4!\, x^{-6} = 5!\, x^{-6}$, $\ldots$,

$f^{(n)}(x) = (n+1)!\, x^{-(n+2)}$ for $n = 0, 1, 2, 3, \ldots$;

~~then~~

$a_0 = \frac{f(1)}{0!} = \frac{1}{1} = 1$, $a_1 = \frac{f'(1)}{1!} = \frac{-2}{1} = -2$,

$a_2 = \frac{f''(1)}{2!} = \frac{3!}{2!} = 3$, $a_3 = \frac{f'''(1)}{3!} = \frac{-4!}{3!} = -4$,

$a_4 = \frac{f^{(4)}(1)}{4!} = \frac{5!}{4!} = 5, \ldots,$ $a_n = (-1)^n (n+1)$

for $n = 0, 1, 2, 3, \ldots$; then

$$\frac{1}{x^2} = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots$$
$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \cdot (n+1) \cdot (x-1)^n .$$

29) $f(x) = e^x$, $a = 2$:

$f'(x) = e^x$, $f^{(n)}(x) = e^x$ for $n = 0, 1, 2, 3, \ldots$;

so $a_0 = \frac{f(2)}{0!} = \frac{e^2}{1} = e^2$, $a_1 = \frac{f'(2)}{1!} = \frac{e^2}{1} = e^2$,

$a_2 = \frac{f''(2)}{2!} = \frac{e^2}{2!}$, $a_3 = \frac{e^2}{3!}$, $\ldots$, $a_n = \frac{e^2}{n!}$ for

$n = 0, 1, 2, 3, \ldots$; then

$$e^x = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + \cdots$$
$$= e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n .$$

41) $f(x) = \ln(\cos x)$, $a = 0$:

$f'(x) = \frac{1}{\cos x} \cdot -\sin x = -\tan x$,

$f''(x) = -\sec^2 x$; so $a_0 = \frac{f(0)}{0!} = \frac{\ln(\cos 0)}{1}$

$= \frac{\ln 1}{1} = \frac{0}{1} = 0$, $a_1 = \frac{f'(0)}{1!} = \frac{-\tan 0}{1} = 0$,

$$a_2 = \frac{f''(0)}{2!} = \frac{-\sec^2 0}{2} = \frac{-1}{2}$$; then

a.) $P_1(x;0) = a_0 + a_1 x = 0 + 0\cdot x = 0$

b.) $P_2(x;0) = a_0 + a_1 x + a_2 x^2$

$= 0 + 0\cdot x - \frac{1}{2}x^2 = \frac{-1}{2}x^2$

43) $f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$, $a = 0$:

$$f'(x) = \frac{-1}{2}(1-x^2)^{-3/2}\cdot(-2x) = \frac{x}{(1-x^2)^{3/2}},$$

$$f''(x) = \frac{(1-x^2)^{3/2}(1) - x\cdot\frac{3}{2}(1-x^2)^{1/2}\cdot(-2x)}{(1-x^2)^3}$$

$$= \frac{(1-x^2)^{1/2}\cdot[(1-x^2) + 3x^2]}{(1-x^2)^3} = \frac{1-2x^2}{(1-x^2)^{5/2}};$$

then $a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1$,

$a_1 = \frac{f'(0)}{1!} = \frac{0}{1} = 0$, $a_2 = \frac{f''(0)}{2!} = \frac{1}{2}$; then

a.) $P_1(x;0) = a_0 + a_1 x = 1 + 0\cdot x = 1$

b.) $P_2(x;0) = a_0 + a_1 x + a_2 x^2$

$= 1 + 0\cdot x + \frac{1}{2}x^2 = 1 + \frac{1}{2}x^2$