Math 25
Vogler

Some Basics for Proofs

Sets

Defn 1) A set S is a collection of objects.
2) $x \in S$ means x is an element of S.
3) $x \notin S$ means x is not an element of S.
4) $\emptyset$ represents the empty set (i.e. one with no elements)

Note: For this class, we'll be working with sets of numbers.

Ex Here are a collection of sets we'll be dealing with.
a) $\mathbb{N} := \{1, 2, 3, \ldots\}$ are the natural numbers.
b) $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are the integers.
c) $\mathbb{Q} := \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$ are the rational numbers.
d) $\mathbb{R}$ is the set of all real numbers (including decimals)
e) $\mathbb{I}$ is the set of all irrational numbers.

Notes: 1) $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ 2) $\mathbb{I} = \mathbb{R} - \mathbb{Q}$

Quantifiers Let $P(x)$ be a proposition involving x.
Ex $P(x) = (|x| < 7)$
Defn 1) $(\forall x)\, P(x)$ means 'for all x, $P(x)$'
2) $(\exists x)\, P(x)$ means 'there exists an x such that $P(x)$'
3) $(\exists! x)\, P(x)$ means 'there exists a unique x such that $P(x)$'

Negation of Quantifiers $\sim$ is the negation of a proposition.

Ex If $P(x) = (3x = 9)$, then $\sim P(x) = (3x \neq 9)$

Lemma 1) $\sim(\forall x)P(x)$ is equivalent to $(\exists x)(\sim P(x))$

2) $\sim(\exists x)P(x)$ is equivalent to $(\forall x)(\sim P(x))$

Notes a) By (1), we can disprove $(\forall x)P(x)$ by finding an x (at least one) for which $P(x)$ is false, such an x is called a counter example.

b) We cannot prove $(\forall x)P(x)$ by giving an example(s).

Subsets

Defn Let A and B be sets. We say A is a subset of B, written $A \subseteq B$, if and only if (iff) every element of A is an element of B. Using symbols, we have $(\forall x)(x \in A \Rightarrow x \in B)$ or $\forall x \in A \Rightarrow x \in B$

Ex 1) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Defn Sets A and B are equal, written $A = B$, if

$$A \subseteq B \text{ and } B \subseteq A$$

Ex 1) $\{n : n \text{ is even}\} = \{m+1 : m \text{ is odd}\}$

Conditionals

1) The conditional $P \Rightarrow Q$ means 'if P then Q'

Note: A conditional is true only when P forces Q to be true.

2) The biconditional $P \Leftrightarrow Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$.