Defn
Let \( \{s_n\} \) be a sequence of real numbers.

1) \( \{s_n\} \) is nondecreasing if \( s_n \leq s_{n+1} \) \( \forall n \in \mathbb{N} \).

2) \( \{s_n\} \) is nonincreasing if \( s_n \geq s_{n+1} \) \( \forall n \in \mathbb{N} \).

3) \( \{s_n\} \) is monotone (or monotonic) if it's nondecreasing or nonincreasing.

Note: If \( \{s_n\} \) is nondecreasing (or nonincreasing), then \( s_n \leq s_m \) (or \( s_n \geq s_m \)) when \( n \leq m \).

Thm (10.2)
All bounded monotone sequences converge.

Thm (10.4)
1) If \( \{s_n\} \) is an unbounded nondecreasing sequence, then
   \[ \lim_{n \to \infty} s_n = \infty \]

2) If \( \{s_n\} \) is an unbounded nonincreasing sequence, then
   \[ \lim_{n \to \infty} s_n = -\infty \]

Note: This fact combined with Thm 10.2 tells us monotonic sequences must go 'somewhere' (either a finite number or \( \pm \infty \)). Refer to Cor. 10.5 for more details.
Defn
Let \( \{s_n\} \) be a sequence in \( \mathbb{R} \). Then, we define

1) the **limit superior** (a.k.a. \( \lim \sup \)) is
\[
\lim_{n \to \infty} s_n = \lim \sup \{s_n : n > N\}
\]

2) the **limit inferior** (a.k.a. \( \lim \inf \)) is
\[
\lim_{n \to \infty} s_n = \lim \inf \{s_n : n < N\}
\]

Notes:

a) One can think of \( \lim \sup / \lim \inf \) as the supremum/infimum (i.e. least upper bound/greatest lower bound) of **ALL tails** of the sequence.

b) \( \lim_{n \to \infty} s_n \) might not equal \( \sup \{s_n : n \in \mathbb{N}\} \) (i.e. the least upper bound of the set containing all the sequence elements), but we can show \( \lim_{n \to \infty} s_n \leq \sup \{s_n : n \in \mathbb{N}\} \). A similar statement applies to \( \lim \inf \).

c) These concepts will be developed further in Section 12.

Thm (10.7)
Let \( \{s_n\} \) be sequence in \( \mathbb{R} \).

1) If \( \lim_{n \to \infty} s_n = L \), then \( \lim \inf_{n \to \infty} s_n = \lim_{n \to \infty} s_n = \lim \sup_{n \to \infty} s_n = L \).

2) If \( \lim \sup_{n \to \infty} s_n = \lim \inf_{n \to \infty} s_n \), then \( \lim_{n \to \infty} s_n = \lim \inf_{n \to \infty} s_n = \lim \sup_{n \to \infty} s_n \).

Note: \( L \) can be finite or \( \pm \infty \)

Defn
A sequence \( \{s_n\} \) is a **Cauchy sequence** if

\( \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n > N \implies |s_m - s_n| < \varepsilon \).

Note: The negation is

\( \exists \varepsilon > 0, \forall N \in \mathbb{N} \text{ with } \exists m, n > N \text{ with } |s_m - s_n| \geq \varepsilon \).

Thm (10.11)
A sequence converges if and only if it is Cauchy.

Note: This is a very useful fact because it allows us to show a sequence converges or diverges without knowing the limit point.